Related Rates

We now look at an important application of implicit differentiation—Related Rates. *Related Rates*, as the name suggests, refers to relating the rates of change of several related quantities. We say two, or more, quantities are related if there's an equation connecting them. As it turns out, if two or more quantities are related, then their rates are related, and we can find the relationship between their rates using implicit differentiation. The goal is to figure out the rate of change of one quantity if the rates of change of the other quantities are known (or can be known in theory, with *relative* ease—pun intended). It is also useful to recall that the word "rate" means that the independent variable is usually *time*, and is often denoted t.

Every related rates problem you will encounter in this course can be solved via the following method. Before going through the method, let's figure out when we'd need to use it in the first place!

How to know when you're looking at a Related Rates problem

You know you're looking at a related rates problem if:

- You see a word problem telling you about two or more related quantities and you are given the rate of change of one (or several) of the quantities.
- You are then asked about the rate of change of another quantity.

Once you know you're looking at a related rates problem, employ the following method.

The Method of Related Rates

1. **Read the problem carefully!** Read it again. Did you read the problem? Read it! The goal it to pick out useful information that will help you with steps 2, 3, and 4. So you're looking for descriptive information that can help you draw a diagram, if needed, figure the rates of change and the value of the quantities at the point of interest.
2. **If possible/necessary[1], draw a diagram. Label the quantities that are changing with variables and the quantities that are not changing by their constant values.** Think of the variables as functions of time. As we will see, some quantities are constant, they do not change. When describing these quantities, use their value. Otherwise, use a variable.
3. **Write down the given information** in regards to the values of any rates that are known. Also **write down what you want to find**, and with what conditions. This will maintain your focus, grasshopper, as well as have the known values explicitly written down ready for use.
4. **Set up an equation that relates all the quantities under consideration.** If you drew a diagram, it will usually come in handy here. For instance, if your diagram is a triangle, the equation you come up with could be Pythogoras' theorem, or an application of similar triangles, etc. So the diagram would suggest what equation you would set up. Remember your geometry! Knowing your geometry will be important here.
5. **Differentiate the equation in step 4 implicitly, with respect to time**. This means that any variable or expression that does not explicitly involve a t will use the chain rule when differentiating (you need to multiply by the primes when differentiating them).
6. **Plug in your knowns and solve for the unknown that you seek.** At first, you may end up with several unknowns. In this case, you should be able to go back to the equation in step 4 to solve for all the unknowns you need in order to solve for the particular unknown you care about. Sometimes, you can also use the geometry suggested by the diagram to eliminate some of the variables in play to make your life easier.

[1] The point of drawing a diagram is for you to be able to visualize the problem well enough in order to come up with an equation (needed in step 4) that relates all the quantities mentioned. If you are given an equation or you don't need a visual aid, you don't need to complete this step.

So those are the steps. Six steps to solve any related rates problem. Before moving on to examples, just a word on geometry.

You are expected to know (as in have memorized and be able to apply) formulas for the perimeter and area of common 2D figures, such as: circles, rectangles (including squares), and triangles, etc. You also need to remember how to find the volume of simple 3D figures, like cylinders, cubes and boxes; also know how to find their surface area. Knowing about triangles (Pythagoras' theorem, SOHCAHTOA, etc.) is also important. For some problems, you will be given the formulas you need (which means step 2 in our process can be ignored).

Examples

1. A right circular cylinder undergoes expansion. Its radius is increasing at a rate of 1 m/s while its height is increasing at a rate of 2 m/s. At what rate is its volume changing when the radius is 1m in length and the height is 3m?

2. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

3. A 15-foot ladder was resting against a vertical wall. Suddenly, and without warning, Batman starts pulling the foot of the ladder along the ground, away from the wall at a rate of 2 ft/s. At what rate is the top of the ladder sliding down the wall when the foot of the ladder is 9 feet from the wall? I'm sure the Dark Knight had his reasons.

4. The volume of a sphere is increasing at a rate of 5 cubic feet per sec. How fast is its radius increasing when the radius is 2 feet in length? The volume of a sphere of radius r is given by the formula $V = \frac{4}{3}\pi r^3$.

5. The side length of a cube is increasing at a rate of 2 in/min. When the side length is 3 in,
   (a) How fast is its surface area changing?
   (b) How fast is its volume changing?

6. The radius of a circle is increasing at a rate of 1 m/min. What is the instantaneous rate of change of the area of the circle at the instant the area is 4π cubic meters?

7. A water tank has the shape of an inverted circular cone and has base radius 3 meters and height 9 meters. It starts out filled with water, but the water is leaking out from the bottom at a rate of 2 m^3/minute. For some odd reason, we want to know how the radius of the remaining water is decreasing when the water-radius is 1 meter. Please determine this rate, so that we may put our minds at ease. Recall that the volume of a circular cone is given by $V = \frac{1}{3}\pi r^2 h$.

8. A man walks along a straight path at a speed of 5 ft/s. A search light is located on the ground 16 ft from the path and is kept focused on the man. He's wearing a hoodie and looks quite suspicious. Also, he could either have a weapon or a pack of skittles in his hand. Can't be too safe. At what rate is the search light rotating when the man is 12 ft from the point on the path closest to the search light?

9. At noon, ship A is 80 miles west of ship B. At this time, both ships set sail. Ship A sails south at a speed of 20 mph, while ship B sails north at a speed of 10 mph. How fast is the distance between the two ships changing at 2pm if both ships maintain constant speed?

10. Superman is cruising at 390 feet per second at an altitude of 5000 feet and he flies directly over Jhevon's house. Assuming Superman maintains altitude and speed, how fast is the distance from the house to Superman changing at the time when Superman is 13,000 feet from the house.

11. (a) A particle is moving along the curve $y = x^3 + 1$ in such a way that its y-coordinate is increasing at a rate of 2 units/sec. At what rate is its x-coordinate changing when $y = 9$?
    (b) At this instant, how fast is the distance between the particle and the point (2,10) changing? Is the particle approaching (2,10) or getting farther from it at this point?

12. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs toward first base with a speed of 24 ft/s.
    (a) At what rate is his distance from second base decreasing when he is halfway to first base?
    (b) At what rate is his distance from third base increasing at the same moment?

P.S. The situations in Related Rates can vary widely, so I suggest you practice more than is required in the homework. You can use other texts as well. Problems will often involve geometry of some sort, but they also may not (see problem 10a, for example). Also be aware of how knowing whether a quantity is increasing or decreasing can change a problem. You can vary some of the problems in this way to see how things would change. For example, in problem 2, we could have made the length increase at 8 cm/s, but the width decrease at a rate of 3 cm/s (how would your solution change in this case?). Another consideration (that may never come up, but better safe than sorry, so I'll mention this anyway) is that the units are consistent. If, for example, in the same problem lengths are mentioned in both m and cm, make sure you convert ALL units of lengths to the same unit before plugging in quantities or making formulas. Sometimes, the unit of the required answer will be indicated, so make sure you convert your units to be compatible with that. So, for example, if a problem ends with something like, "How fast is this length increasing in meters per second?", then you'd make sure that all your lengths are converted to meters and all your time units are converted to seconds before doing anything.