MATHE 4800C FOUNDATIONS OF ALGEBRA AND GEOMETRY CLASS NOTES FALL 2011

MATTHEW AUTH, LECTURER OF MATHEMATICS

1. Foundations of Euclidean Geometry (Week 1)

During the first weeks of the semester we will discuss the content of Greek mathematics described in the "Elements". We will not study the "Elements" directly, although I do encourage you to glance at it from time to time (I've included a link to a translation of the "Elements" on our class site). Instead we will study Kiselev's "Planimetry" a relatively modern classroom friendly geometry book that follows the spirit of the "Elements" but does not adhere rigorously to the axiomatic approach.

The "Elements" were written by the Greek mathematician Euclid in approximately 300 B.C. Learning the foundations of geometry by reading the "Elements" can be intimidating because the quantity and quality of its theorems. One if often left wondering how any one person was able to discover them all. Indeed, Euclid did not prove most of the theorems from the "Elements" himself. He collected them from many generations of mathematicians. Unfortunately for anyone reading the elements this makes it is impossible to determine what motivated the geometers who first thought about the results contained inside. Little is known of Euclid's life and motivations as well.

What is most impressive about Euclid's "Elements" is the manner in which Euclid presented the theorems. He did not lay them down haphazardly but instead arranged them in such an intricate fashion that the "Elements" are considered one of man's most profound and far-reaching written achievements.

Euclid started by assuming five supposedly self-evident geometric facts without proving them. He called these facts axioms (or postulates) to distinguish them from theorems which were not necessarily self-evident and required proof. An example of an axioms is "two points determine a line". Euclid then combined the axioms with logic (his common notions) and started to prove theorems. Furthermore, after proving a theorem, Euclid used that theorem together with the axioms and common notions in order to prove still more theorems . . . This is Euclid's method, the axiomatic method. (I suggest that you read the axioms and common notions in Book 1 of Euclid's text. There is a link to a translated version on our website. You should be warned that this translation uses the word proposition instead of the word theorem used in our translation of "Planimetry")

Using the axiomatic method Euclid quickly and efficiently established theorems which are not at all self-evident. Whenever you find a statement that is not evident while reading Euclid or a similar geometry text like Kiselev's "Planimetry" you can

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repeatedly skim back in the text to verify the theorems which are used to prove the confusing statement until you are satisfied with the explanation. If you skim back far enough, you return to the axioms whose truth you agreed to accept without verification at the beginning of the text.

I believe that it is appropriate for all students to study the axiomatic method, even students who may not be interested in geometry. By working within the axiomatic system introduced in the "Elements" students learn a method used to organize information coherently and efficiently in many subjects. From time to time economists, physicists, and others adopt the blueprint, provided first by Euclid, for using logic to make complex predictions (theorems) from a limited number of assumptions (axioms). This is because the human mind is limited in its ability to store and digest large collections of complex facts. A person schooled in the axiomatic method need only remember the axioms as well as the logic and then, if sufficiently talented, can reconstruct the subject herself.

2. The Integers and the Decimal System (Week 2)

While we continue to study the foundations of geometry, we will begin studying the foundations of algebra this week. The goal is to use the five laws (or axioms) of arithmetic found on p.5 of Courant's text to describe the multiplicative algorithm

$$
\begin{array}{ccccc}
 & & 3 & 8 & 4 \\
 & \times 1 & 5 & 6 \\
\hline
 & 2 & 3 & 0 & 4 \\
1 & 9 & 2 & 0 & \\
3 & 8 & 4 & & \\
\hline
5 & 9 & 9 & 0 & 4
\end{array}
$$

and the division algorithm

$$
\begin{array}{r}
949 \\
13\overline{)12345} \\
\underline{117} \\
64 \\
\underline{52} \\
125 \\
\underline{117} \\
8
\end{array}
$$

taught in middle school.

In order to understand how this works it can be helpful to work out a few multiplication and a few division problems using an alternate system to the familiar decimal system. The reason for this is that it is more difficult to rely on memorized habits which can hinder true understanding when using an alternate system. I have therefore assigned some problems using the binary (dyadic) system in your weekly homework.

I will not give as much guidance this week. I hope that you can use the principles we have established while studying the foundations of geometry as well as your experience doing basic algebra to describe some of the foundations of algebra.

3. The Congruence Tests SAS, ASA, and SSS (Week 3)

This week we describe how to prove the three congruence tests for triangles using the foundational principle, "Two geometric figures are equivalent if one can be superimposed onto the other by a rigid motion". This congruence principle is so appealing because it is intuitive. However there is a price to pay for elegance. It is difficult to provide purely logical arguments to prove the Congruence Tests as theorems. Even Euclid slipped up while proving the Congruence Tests by making assumptions about rigid motions that do not follow convincingly from the axioms. This can all be fixed quite easily today using linear algebra but we will not study linear algebra at this time.

Instead you should pay particular attention to the arguments given in sections 5 and 6 of Chapter 1 of Kiselev's text. Like Euclid's proofs these arguments about superimposing shapes are not as convincing as other arguments in Kiselev's text.

4. Parallel Lines (Week 4)

We introduce parallel lines and parallelograms this week. Along the way we prove the famous theorem that sum of the angles of a triangle is $180°$. As we progress through Kiselev's text, the theorems and exercises become less intuitive; we may even find ourselves questioning their plausibility. Fortunately at this stage the proofs become more convincing.

Once we have accepted the Congruence Tests as proved we can often use them to prove less intuitive assertions. One other result that is often used in these sections is the Exterior Angle Theorem for triangles. It is proved using SAS. You should familiarize yourself with it as well as how it is used to prove results about parallel lines.

5. Rational and Irrational Numbers (Week 5)

We return to study the foundations of algebra in Courant's text this week. While you read the text pay particular attention to how and why the number concept was developed, from the natural numbers, to the integers, to rationals and then irrationals. This may all seem quaint or obvious because you are familiar with algebraic computations. Do not be fooled. The development of the number concept will not seem obvious to your students. Moreover, as you work through the weekly homework assignment, you may find that things you thought were obvious are not.

Today we use many powerful computational techniques like long division in order to manipulate numbers. These techniques open mathematical avenues to students that were at one time closed. However you should not confuse the techniques with true a understanding of numbers.

I will provide two examples of how understanding a technique is insufficient to understand numbers.

The first concerns the rule for adding fractions

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

It is one thing to be able master the technique of adding fractions by completing many exercises; it is another thing to understand the rule itself. Here is a long winded explanation that I do not advise using with young students.

At some point students in grade school begin to solve equations like

$$3 \times 12 = x$$

for x. These students have previously memorized their multiplication tables so finding the solution is routine. They just need to get used to the letter x appearing in an equation. After awhile, when they are properly motivated, the teacher rearranges the equation so that it becomes

$$3 \times x = 12.$$

Again the students can solve it by returning to their multiplication tables and inverting the operation. (Note that they are not required to independently memorize a division table; that would be too much work when they can divide using the multiplication table. Math may be difficult at times but it sure is efficient.)

However when the teacher rearranges the equation again

$$x \times 12 = 3,$$

students can no longer solve it by returning to the multiplication table. They're stuck. The students can either state that such equations have no solutions or they can create a new number $x = \frac{1}{4}$. It is easier, of course, to claim that these "bad" equations have no solutions and be done with it. This idea however is quickly abandoned when one realizes that it is often necessary to divide quantities. The students are then resigned to accept $\frac{1}{4}$ as a number.

There is a steep price to pay for doing this. Students cannot accept $\frac{1}{4}$ alone. They must accept every expression of the form $\frac{a}{b}$ as a number, a rational number, when a and $b \neq 0$ are integers. They must subsequently learn how to add, subtract, multiply, and divide these rational numbers in such a way that they do not break any of the original laws of algebra. The key idea is that the algebraic rules for operating with integers should not be broken.

This takes a lot of time. A good part of the middle school math curriculum is dedicated to learning how to manipulate fractions. I have assigned a homework problem asking you finish this explanation and to explain how one attempt to add fractions breaks the laws of algebra.

The second example is about exponents. Suppose that a student approaches you and asks you why does $7^0 = 1$? Here is one answer that you could give that uses the laws of algebra. You could explain to the student that $7^5 = 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$ and $7^3 = 7 \cdot 7 \cdot 7$ by definition. So

$$7^5 \cdot 7^3 = (7 \cdot 7 \cdot 7 \cdot 7 \cdot 7) \cdot (7 \cdot 7 \cdot 7) = 7^8.$$

After playing around with examples like these for a little while you can convince the student that they are examples of a general multiplication law: $a^m \cdot a^n = a^{m+n}$. This is all easy to follow. Then you show the student a few examples like

$$\frac{7^5}{7^3} = \frac{7 \cdot 7 \cdot 7 \cdot 7 \cdot 7}{7 \cdot 7 \cdot 7} = \frac{7 \cdot 7 \cdot 7}{1} = 7^2.$$

You convince her that there is also a division law: $\frac{a^m}{a^n} = a^{m-n}$. Finally you ask what is $\frac{7^3}{7^3}$? Clearly $\frac{7^3}{7^3} = 1$. However, using the division law $\frac{7^3}{7^3} = 7^{3-3} = 7^0$. There are two choices at this point. She either must state that $7^0 = 1$ or she must throw out the division law. But we don't want to throw out the division law; we want it to be true. The basic examples leading to the division law have a link to reality. So $7^0 = 1$.

6. Circles

In addition to this weeks readings you may find it helpful to play around with some geometrical software like sketchpad (costs money) or geoGebra (free). I will create a link to geoGebra on our website.

7. Isometries (Week 10)

During the last two weeks of the course we will study rigid motions (or as we have called them: superpositions). Rigid motions have already played a big role in our course. In the first week we defined two geometric shapes to be congruent if there is a rigid motion from one to the other.

In the following we will denote the xy-plane by $\mathbb{R}^2$.

Definition. A *mapping* F of the plane $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a function whose domain and range are both points $P(x, y)$ in the plane $\mathbb{R}^2$.

Example.
(1) $T_{(1,2)}$, Translation by the vector $(1, 2)$ is a map of the plane.
(2) m, mirror reflection over the x-axis is a map of the plane.
(3) R_{60°, rotation by 60° counterclockwise about the origin is a mapping of the plane.
(4) p_x, projection onto the x-axis is a mapping of the plane.

Definition. An *isometry* (also known as a *rigid motion*, or in our class as a *superposition*) is a mapping of the plane that preserves distance, i.e. a mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ in which for any points P and Q in the plane $d(F(P), F(Q)) = d(P, Q)$.

Example. All the above examples of mappings are rigid motions except p_x because p_x maps $P(2, 3)$ and $Q(2, 11)$ to $p_x(P) = p_x(Q) = (2, 0)$. Thus $d(p_x(P), p_x(Q) = 0 \neq 8 = d(P, Q)$.

Our goal in studying isometries is to develop an "algebra" of isometries. This may sound strange at first but thinking this way has its advantages. For one, if we can do "algebra" with rigid motions then we will be able to apply some of our knowledge of algebra to geometry problems. This is a powerful idea, and a big part of our course.

The first thing we want to do is to be able to "multiply" isometries, whatever that may mean. We do this by composing functions. It is not hard but you should practice with the homework problems in order to get good at it.

Definition. Two isometries of the plane F_1 and F_2 can be combined to form a new isometry F_1F_2 so that $F_1F_2(P) = F_1 \circ F_2(P) = F_1(F_2(P))$.

Exercise. Given isometries F_1 and F_2. Prove that the mapping F_1F_2 defined above is truly an isometry.

Example.
(1) $R_{60^\circ}R_{45^\circ} = R_{105^\circ}$.
(2) $T_{(5,2)}T_{(11,-6)} = T_{(16,-4)}$.
(3) $R_{90^\circ}T_{(4,0)} = R_{S,90^\circ}$ when $R_{S,90^\circ}$ is a rotation of 90° about the point $S(-2, 2)$.

The reason that we call composition of rigid motions "multiplication" and not "addition" is that traditionally "additive" operations are reserved for commutative operations. Compositions of isometries is not commutative as can be seen by reversing the order of the last example. $T_{(4,0)}R_{90^\circ} = R_{Q,90^\circ}$ when $Q(2, 2)$.

So $R_{90^\circ}T_{(4,0)} \neq T_{(4,0)}R_{90^\circ}$. You should therefore be careful when working with the algebra of isometries; the algebra of isometries is different from our familiar algebra.

The following theorem which will be proved in class can be very helpful.

theorem 1. An isometry is completely determined by where it takes three nonlinear points.

This theorem can be used for instance in determining that $R_{90^\circ}T_{(4,0)} = R_{S,90^\circ}$ when $R_{S,90^\circ}$ is a rotation of 90° about the point $S(-2,2)$. First verify that $R_{90^\circ}T_{(4,0)}(S) = S$. In this case we say that S is a *fixed point* of the rigid motion $R_{90^\circ}T_{(4,0)}$. Once we have a fixed point, we can judiciously pick two other points, in this case say $P(-2,0$ and $Q(0,2)$, in the plane to verify that $R_{90^\circ}T_{(4,0)}$ is indeed a rotation. Try to do this.

Even though multiplication of isometries is not the same as multiplication of numbers there are a few more similarities that ought to be mentioned. The first is the existence of a multiplicative identity element. In grade school we learn that there is a special number 1 that when multiplied by any other number does not change the other number, i.e $1 \cdot 16 = 16$. There is an analogous rigid motion $I(P) = P$, called the identity.

Example. $IR_{90^\circ} = R_{90^\circ}I = R_{90^\circ}$

After we have I and multiplication we can talk about dividing isometries.

Definition. Given an isometry F of the plane. An isometry G is called the *inverse* of F if $GF = FG = I$. When this is so, we write $G = F^{-1}$ (or we could equally write $F = G^{-1}$.

Example. (1) $T^{-1}_{(1,2)} = T_{(-1,-2)}$.
(2) $R^{-1}_{130^\circ} = R_{230^\circ} = R_{-130^\circ}$
(3) $m^{-1}_{y=x} = m_{y=x}$ when $m_{y=x}$ is the mirror reflection about the line $y = x$.

Finally, after we learn to multiply, we try to factor isometries like numbers. Eighteen can be factored into primes as $18 = 3^2 \cdot 2$ just like the rotation $R_{S,90^\circ}$ by 90° about the point $S(-2,2)$ can be factored into a rotation about the origin multiplied by a translation $R_{S,90^\circ} = R_{90^\circ}T_{(4,0)}$. We may think that it is possible to factor an isometry uniquely into primes but it is not clear how to do this as the following theorems indicate. The theorems will be proved in class.

theorem 2. Every isometry F of the plane can be factored as $F = T_vR$ when T_v is a translation by vector v, and R is some isometry that fixes the origin $O(0,0)$.

We can still do better, and learn more, by factoring the rigid motion R that showed up in the previous theorem.

theorem 3. Every isometry F of the plane can be factored into as $F = T_vR_\alpha m$ or as $F = T_vR_\alpha$ where T_v is a translation by vector v, R_α is a rotation about the origin by α, and m is a mirror reflection about the x-axis.

With this theorem as well as our knowledge of the complex numbers, we can write formulas for rigid motions as we write formulas for functions in high school like $f(x) = 3x^2+1$ where x is the independent variable. The rigid motion $F = T_vR_\alpha$ can be written as $F(z) = (\cos\alpha + i\sin\alpha)z + v$ when z is the independent complex number or point in the plane.

Example. The rigid motion $R_{90^\circ}T_{(4,0)}$ can be written as $R_{90^\circ}T_{(4,0)}(z) = (\cos 90^\circ + i\sin 90^\circ)(z+4+0i) = i(z+4)$. Use this formula to find the fixed points of $R_{90^\circ}T_{(4,0)}$ i.e. the points z so that $R_{90^\circ}T_{(4,0)}(z) = z$ or with our formula $i(z+4) = z$. You should find $z = -2+2i$ which is the point $S(-2,2)$. This is one way to find that $R_{90^\circ}T_{(4,0)} = R_{S,90^\circ}$.

The next theorem is a bit harder but you should be able to prove a version of it.

theorem 4. Every rigid motion F of the plane can be factored as the product of three reflections m_1, m_2 and m_3 i.e. $F = m_1m_2m_3$.

You should also think about the following facts (they are not easy to verify geometrically):

(1) The composition (multiplication) of two rotations $R_{P,\alpha}R_{Q,\theta}$ is, in general, another rotation $R_{K,\alpha+\theta}$. When is the composition of two rotations is a translation?
(2) The composition of two mirror reflections is a rotation or a translation.
(3) The composition of a translation and a rotation is a rotation with the same angle and, in general, a new center.
(4) The composition of a mirror reflection with a rotation is a mirror reflection.

Department of Mathematics, CCNY, NAC 6-288, New York, NY 10031
E-mail address: mauth@ccny.cuny.edu