Curve Sketching

We've done most of the legwork needed for this section. Now let's put it together to sketch some curves!

The steps (to sketch a given curve $y = f(x)$):

1. **Observe/note the domain of f.** This might come in handy.
2. **Find the x- and y-intercepts.** Recall, if they exist, we find the x-intercept(s) by setting $y = 0$ and solving for x. We find the y-intercept by setting $x = 0$ and solving for y.
3. **Find the asymptotes.** If asymptotes exist, we need to know where they are. There are two kinds that are of main concern to us: Vertical asymptotes and horizontal asymptotes.[1]

   For horizontal asymptotes: find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. If either of these give you a finite number (call it L), then $y = L$ is a horizontal asymptote. *Note: it is possible to have two distinct horizontal asymptotes, so you MUST check both limits!*

   For vertical asymptotes: you must find a number a, such that $\lim_{x\to a^+} f(x) = \pm\infty$ and/or $\lim_{x\to a^-} f(x) = \pm\infty$. If such a number exists, then $x = a$ is a vertical asymptote. *Note: a function can have several vertical asymptotes as well.* For rational functions, this occurs when the denominator is zero (AND THE NUMERATOR IS NOT ZERO AT THE SAME TIME!)[2]. For other general functions, you may need to be more creative. But checking where a division by zero is in danger of occurring is a good way to start.

   Asymptotes are represented as broken lines on the graph. Also note, it is possible for f to cross horizontal asymptotes, but it canNOT cross vertical ones.

4. **Find the intervals of increase and decrease and locations of (local) maximums and minimums, if they exist.** Remember, $f'(x)$ and the first derivative test will help you with this.
5. **Find the intervals of concavity and locations of any inflection points, if they exist.** Remember, $f''(x)$ and the concavity test on a number line will help you with this.[3]
6. **Sketch the graph!** With all the above information at your disposal, you will have all you need to draw the graph of $f(x)$. The above steps will tell you the important points, and show you what skeleton your graph will have to fit.

Rookie mistakes students make: Obviously there are the mechanical mistakes with algebra, etc. Forgetting the steps is another rookie mistake (that practicing a lot of problems will take care of—know the steps and do them in the same order every time). The biggest mistake in my experience is students ignoring anomalies! For example, I've seen kids find intercepts and say that there are two intercepts, yet they draw a graph and it will have one intercept, or three. Obviously either your graph or one of your calculations is wrong in this case. Check your work! Don't ignore it and move on and submit your answer

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[1] There are asymptotes known as "slant" and "oblique", but we don't care about those in Math 205.

[2] If both the numerator and denominator of a rational function are zero for the same x-value, it means you can factor both and "cancel" the bad factor. However, since the original function had that factor in the denominator, you must avoid the point. This will result in a "hole", not a vertical asymptote.

[3] Also remember, to find the location of a point, $f(x)$ will help you. Use it to find the y-value of the coordinates.

like nothing is wrong with it! Another rookie mistake is students not remembering which function to use: *Should I use f to compute this? f' or f''?* KNOW the table that I gave you in class. Know when to use which function. Making such a mistake is inexcusable. If you use the derivative to find the y-value of a coordinate, or do something equally silly, there will be no mercy for you! (First) derivatives help you to find where the function is increasing, decreasing or the <u>x-values</u> of points where max's or min's occur. Know what a derivative helps you with. Know what the original function or the second derivative helps you with as well!

Examples: Find all the important features listed above and sketch the graph of the following:

(a) $f(x) = \frac{1}{x^2}$ (b) $f(x) = x^4 - 4x^2$ (c) $f(x) = x^4 - 4x^3$

(d) $f(x) = -3x^5 + 5x^3$ (e) $f(x) = 2x^{5/3} - 5x^{4/3}$ (f) $f(x) = \frac{x^2+1}{x^2-4}$

(g) $f(x) = x^3 + 2x^2 + x$ (h) $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 5$ (i) $f(x) = x^3 - x^2 - x$

Note that (a) and (f) have asymptotes.

More examples—of the format you will likely see in a test or final: Consider the given function $f(x)$. The formulas for $f'(x)$ and $f''(x)$ are given and need not be verified. Find all intercepts, asymptotes, relative extrema, intervals of increasing and decreasing, intervals of concavity and inflection points (if they exist) for the function f. Use this information to sketch the graph of $f(x)$, and indicate on your graph all the features requested above.[4]

(The following problems were taken from Math 201 past finals. You will be practicing the ones from the Math 205 finals on your own):

(a) $f(x) = \frac{x^2}{x^2-4}$ given that $f'(x) = \frac{-8x}{(x^2-4)^2}$ and $f''(x) = \frac{8(3x^2+4)}{(x^2-4)^3}$ (Taken from Spring 2010 final).

(b) $f(x) = \frac{4x}{x^2+1}$ given that $f'(x) = \frac{4(1-x^2)}{(x^2+1)^2}$ and $f''(x) = \frac{8x(x^2-3)}{(x^2+1)^3}$ (Taken from Fall 2010 final).

(c) $f(x) = \frac{x^2-9}{(x-1)^2}$ given that $f'(x) = \frac{2(9-x)}{(x-1)^3}$ and $f''(x) = \frac{4(x-13)}{(x-1)^4}$ (Taken from Fall 2009 final).

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[4] Sometimes you will be given the first and second derivative, but not always. The point of these problems is not to test your ability to take derivatives (that will be tested in many other problems on a test or final), the point of this problem is to test whether you know what the different derivatives mean and what they can tell you about a graph and whether you can put all this info together and produce a correct graph for the function. Of course, if the derivatives are not given, you're expected to find them on your own. This is less typical, but be prepared to do it just in case. And make sure you include everything that is asked for! If you skip the step where you found the intercepts when intercepts were specifically asked for, for example, you will lose points.