Math 20300
Calculus III

Lesson 34
Taylor and Maclaurin Series

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Taylor and Maclaurin Series

Which functions can be represented by power series, and how do we find these power series?

Suppose we have a function $f(x)$ that can be represented by a power series, ie

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

$$= a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

for some interval of convergence.

Plug in $x=c$: $f(c) = a_0 + \underbrace{0+0+0+\cdots}_{\text{all zero}}$

$$\therefore \ f(c) = a_0 .$$

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

$$f'(c) = a_1 + \underbrace{0 + 0 + \cdots}_{\text{all zero}} \quad \therefore \quad f'(c) = a_1.$$

$$f''(x) = 2a_2 + 3(2)a_3(x-c) + 4(3)a_4(x-c)^2 + \cdots$$

$$f''(c) = 2a_2 \quad \Rightarrow \quad a_2 = \frac{f''(c)}{2}$$

$$f'''(x) = 3(2)a_3 + 4(3)(2)a_4(x-c) + \cdots$$

$$f'''(c) = 3(2)a_3 \quad \Rightarrow \quad a_3 = \frac{f'''(c)}{3!}$$

and $f^{(n)}(c) = n!\,a_n \Rightarrow a_n = \dfrac{f^{(n)}(c)}{n!}$

n^{th} derivative

$$\therefore \; f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots$$

pagebreak pagebreak pagebreak pagebreak pagebreak pagebreak pagebreak pagebreak pagebreak or we can write $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

So, <u>if</u> f has a power series representation,

it must be $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

for some interval of convergence.

This is called the <u>Taylor series</u> expansion of f at $x = c$ (centered at $x = c$, about $x = c$)

When $c = 0$ (centered at $x = 0$) this is called the <u>Maclaurin series</u> expansion of f, ie

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n .$$

So far, if f has a power series expansion, it must be the Taylor series expansion.

But, under what circumstances does f have a power series expansion?

ie, under what circumstances

$$\text{does } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \text{ ?}$$

when are these guaranteed to be equal (for some interval of x-values)?

Well, when the series converges, that is equivalent to the sequence of partial sums converging.

let $$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n$$ N^{th} degree Taylor polynomial for $f(x)$

and $$R_N(x) = \sum_{n=N+1}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$ Remainder term for the N^{th} degree Taylor polynomial for f.

so if $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n,$$

$$f(x) = T_N(x) + R_N(x)$$

and $$\lim_{N\to\infty} T_N(x) = f(x)$$

sequence of partial sums

Then $$R_N(x) = f(x) - T_N(x)$$

and $$\lim_{N\to\infty} R_N(x) = \lim_{N\to\infty}\left(f(x) - T_N(x)\right)$$

$$= f(x) - \lim_{N\to\infty} T_N(x) =$$

$$= f(x) - f(x) = 0.$$

$\therefore \lim_{N\to\infty} R_N(x) = 0$ equivalent to saying

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

(we're getting closer to an answer...)

Taylor's Formula: If f has $N+1$ derivatives on interval I around $x=c$, then for each $x \in I$, $\exists z$ strictly between x and c $\ni$

$$R_N(x) = \frac{f^{(N+1)}(z)}{(N+1)!}(x-c)^{N+1}$$

not c, as with terms of Taylor series. some z that is between x + c.

Note: z depends on x and N.

Now to answer our question...

To see if $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$, we

look at $R_N(x)$ as given by Taylor's formula,

and see if $\lim_{N\to\infty} R_N(x) = 0$.

if so, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.

(still need to find interval of convergence)

if not, $f(x)$ can not be represented

by a power series.

Note: $\lim_{n\to\infty} \frac{x^n}{n!} = 0$. ← we use This in lots of $R_N(x)$ limits

proof: Consider $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

Ratio test: $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0 \ \forall x.$

$\therefore$ series converges, $\therefore$ sequence $\frac{x^n}{n!} \to 0$.

Ex. for $f(x) = e^x$, prove that $f(x)$ can be represented by a power series, and find the series.

We need $\lim_{N \to \infty} R_N(x) = 0 \ \forall x$ in the interval of convergence of $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.

this will be our power series.

start by finding the series (centered at $x=0$)

for $f(x) = e^x$, $f^{(n)}(x) = e^x \ \forall n$, $f^{(n)}(0) = e^0 = 1 \ \forall n$

so $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is the series.

Interval of convergence:

Ratio test: $\lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$

$$= \lim_{n \to \infty} \left| \frac{x}{n+1} \right| = 0 = L < 1 \quad \forall x$$

so interval of convergence is $\mathbb{R}$ (all reals)

Now, to prove $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\forall x$, we need to show

$$\lim_{N \to \infty} R_N(x) = 0 \quad \text{where } R_N(x) = \frac{f^{(N+1)}(z)}{(N+1)!} x^{N+1}$$

so $R_N(x) = \frac{e^z}{(N+1)!} x^{N+1}$ for some z between x and 0 ← center of series

Note that z also depends on N. so before we can take $\lim_{N \to \infty} R_N(x)$, we need to replace the e^z with something about x.

we know z is between x + 0.

If $x > 0$, then $0 < z < x$

and $e^z < e^x$

so $R_N(x) = \frac{e^z}{(N+1)!} x^{N+1} < \frac{e^x x^{N+1}}{(N+1)!}$

since $x > 0$,

$$0 \leq \underbrace{\frac{e^z}{(N+1)!} x^{N+1}}_{R_N(x)} < \frac{e^x x^{N+1}}{(N+1)!} \to 0 \text{ as } N \to \infty$$

since $\lim_{N\to\infty} \frac{e^x x^{N+1}}{(N+1)!} =$

$$e^x \left(\lim_{N\to\infty} \frac{x^{N+1}}{(N+1)!} \right)$$

$$= e^x (0) = 0 .$$

$\therefore$ by the Squeeze Theorem, $\lim_{N\to\infty} R_N(x) = 0$.

if $x < 0$, $\quad x < z < 0$

and $\quad e^z < e^0 = 1$

so $\quad R_N(x) = \frac{\overbrace{e^z}^{\text{always positive}}}{(N+1)!} x^{N+1} \leq \frac{x^{N+1}}{(N+1)!}$

$$0 \le |R_N(x)| \le \frac{|x|^{N+1}}{(N+1)!} \to 0 \quad \text{as } N \to \infty$$

$$\therefore \quad \lim_{N\to\infty} |R_N(x)| = 0 \quad \text{and} \; \therefore \lim_{N\to\infty} R_N(x) = 0.$$

for $x = 0$, $\quad e^0 = 1 \qquad \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \cdots$

$x = 0, \qquad = 1 + 0 + 0 \cdots$

$\therefore$ The series converges to e^x at $x = 0$.

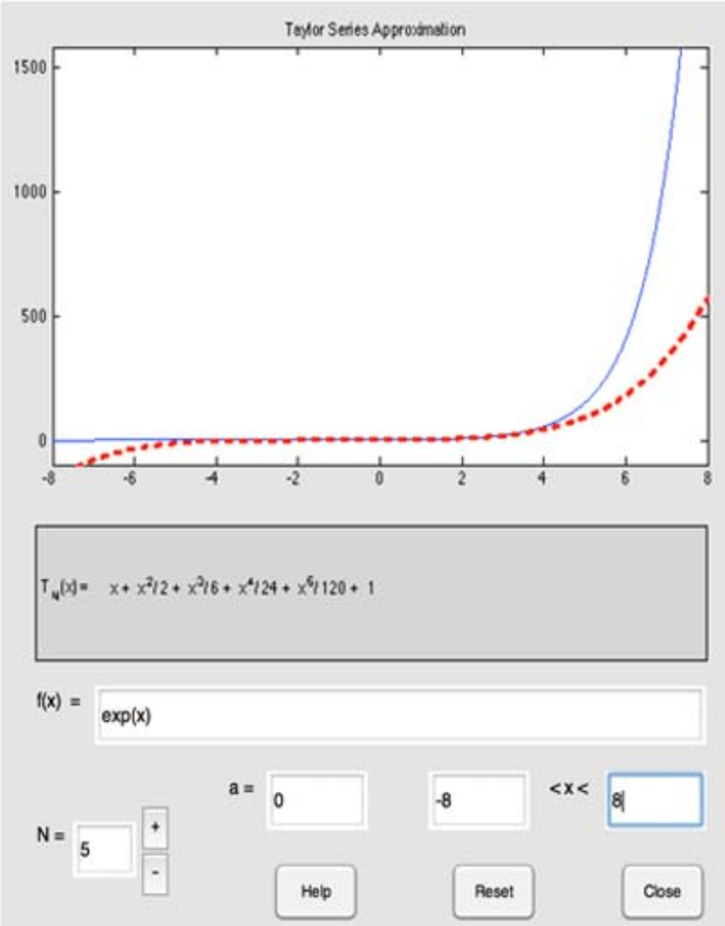
Remember that this means

the sequence of $\underbrace{\text{partial sums of the series}}_{T_N(x) \text{ taylor polynomials}}$

converges to $f(x)$.

Taylor polynomial of degree N:

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n = \sum_{n=0}^{N} \frac{x^n}{n!} \quad \text{here}$$


Taylor Series Approximation
1500
1000
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0
-8 -6 -4 -2 0 2 4 6 8
T_N(x) = x + x^2/2 + x^3/6 + x^4/24 + x^5/120 + 1
f(x) =
exp(x)
a =
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N =
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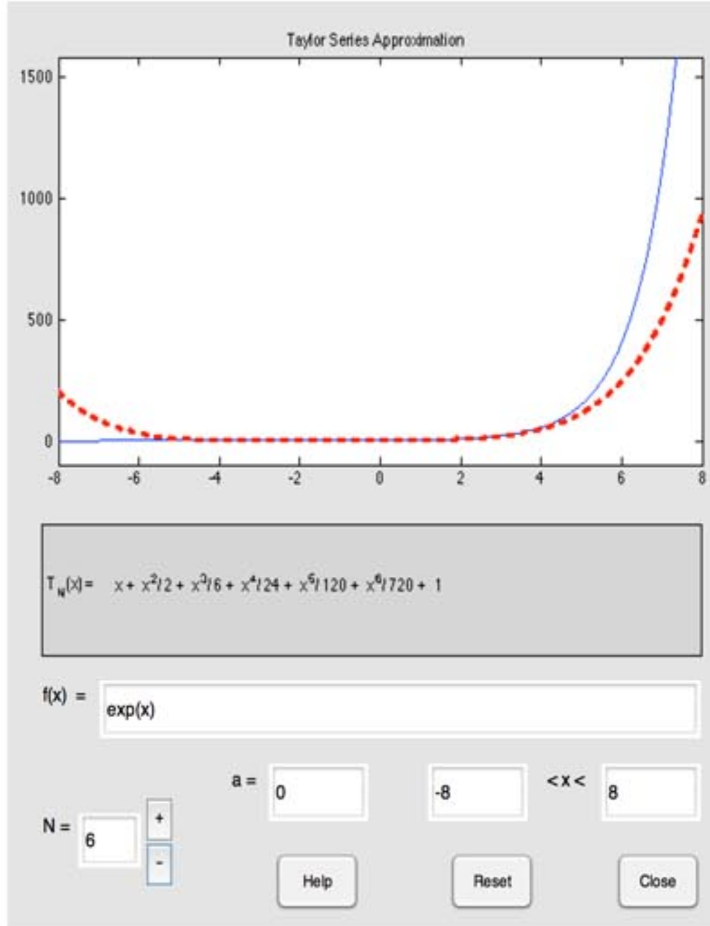
Taylor Series Approximation
1500
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-8 -6 -4 -2 0 2 4 6 8
T_N(x) = x + x^2/2 + x^3/6 + x^4/24 + x^5/120 + x^6/720 + 1
f(x) =
exp(x)
a =
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N =
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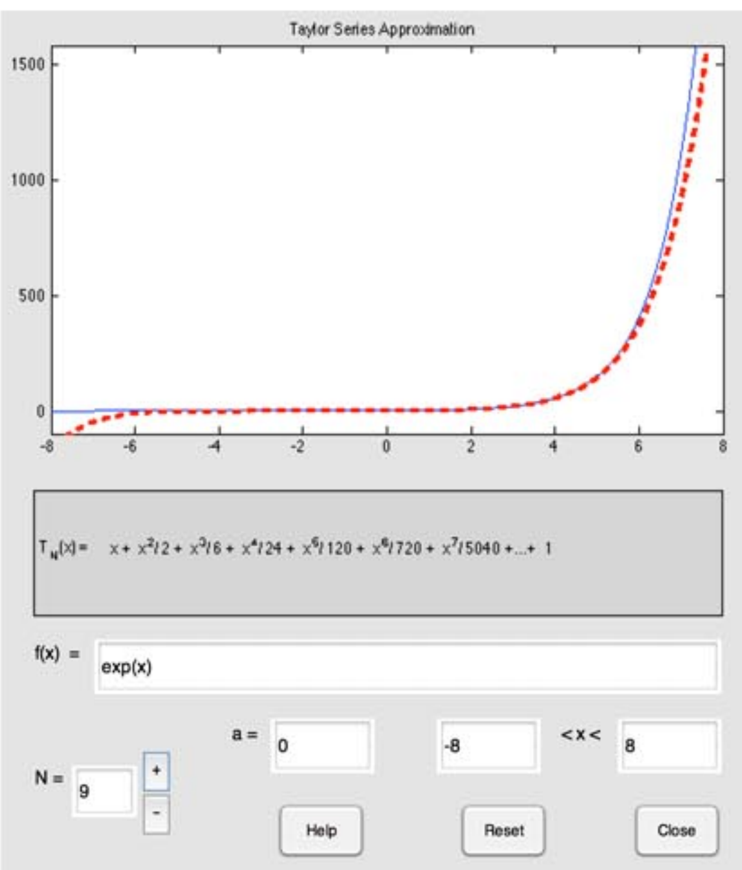
Taylor Series Approximation
1500
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-8 -6 -4 -2 0 2 4 6 8
T_N(x) = x + x^2/2 + x^3/6 + x^4/24 + x^5/120 + x^6/720 + x^7/5040 +...+ 1
f(x) =
exp(x)
a =
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N =
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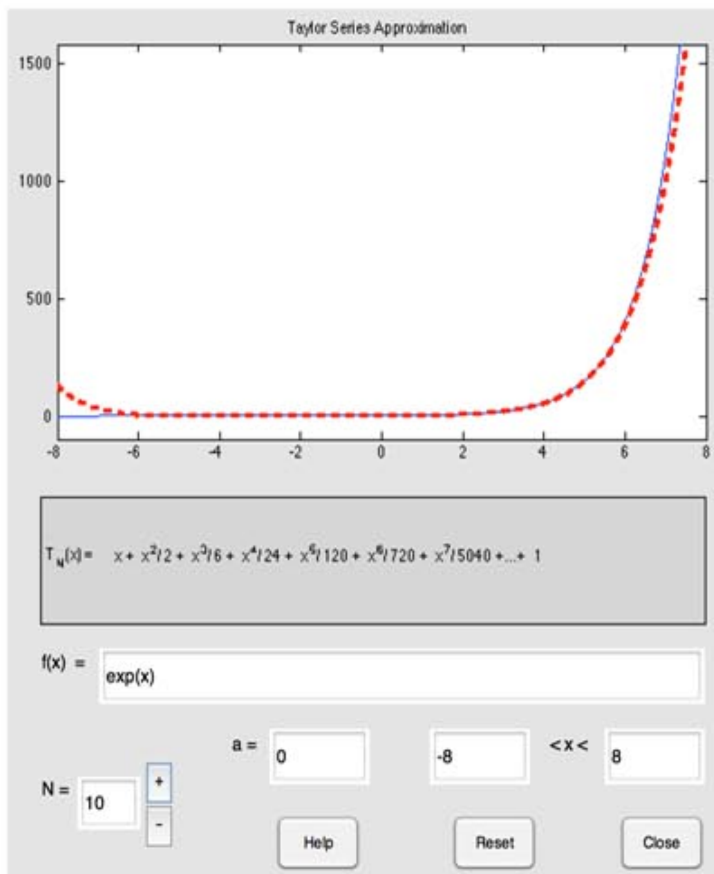
Taylor Series Approximation
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-8 -6 -4 -2 0 2 4 6 8
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f(x) =
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Ex. Prove That $f(x) = \cos x$ has a power series expansion around $x = \frac{\pi}{2}$, and find The series.

Start by finding The series and its interval of convergence

The series will be $\displaystyle\sum_{K=0}^{\infty} \frac{f^{(K)}(\frac{\pi}{2})}{K!}(x-\frac{\pi}{2})^K$

for $f(x) = \cos x = f^{(0)}(x)$ $\qquad f^{(0)}(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

$f'(x) = -\sin x$ $\qquad f'(\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1$

$f''(x) = -\cos x$ $\qquad f''(\frac{\pi}{2}) = -\cos\frac{\pi}{2} = 0$

$f'''(x) = \sin x$ $\qquad f'''(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

$$\therefore \sum_{K=0}^{\infty} \frac{f^{(K)}(\frac{\pi}{2})}{K!}(x-\frac{\pi}{2})^K =$$

$$= -(x-\tfrac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^3}{3!} - \frac{(x-\frac{\pi}{2})^5}{5!} + \frac{(x-\frac{\pi}{2})^7}{7!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$$

to find the interval of convergence,

Ratio test: $$\lim_{n\to\infty} \left| \frac{\left(x-\frac{\pi}{2}\right)^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{\left(x-\frac{\pi}{2}\right)^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\left(x-\frac{\pi}{2}\right)^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{\left(x-\frac{\pi}{2}\right)^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\left(x-\frac{\pi}{2}\right)^{2}}{(2n+3)(2n+2)} \right| = 0 = L < 1 \quad \forall x$$

$\therefore$ series converges $\forall x \in \mathbb{R}$.

Now, to prove it converges to $\cos x$:

need $\lim_{N\to\infty} R_N(x) = 0 \quad \forall x$.

we know $R_N(x) = \dfrac{f^{(N+1)}(z)}{(N+1)!}\left(x-\frac{\pi}{2}\right)^{N+1}$ for some z between x and $\frac{\pi}{2}$.

since $f^{(N+1)}(z)$ is $\pm\cos z$ or $\pm\sin z$ we know $|f^{(N+1)}(z)| \leq 1$.

$$\therefore\ 0 \le |R_N(x)| = \frac{|f^{(N+1)}(z)| \cdot |x - \frac{\pi}{2}|^{N+1}}{(N+1)!} \le \frac{|x-\frac{\pi}{2}|^{N+1}}{(N+1)!} \to 0 \text{ as } N \to \infty.$$

by the Squeeze Theorem $\lim_{N\to\infty} |R_N(x)| = 0 \quad \forall x$

$$\therefore \lim_{N\to\infty} R_N(x) = 0 \quad \forall x$$

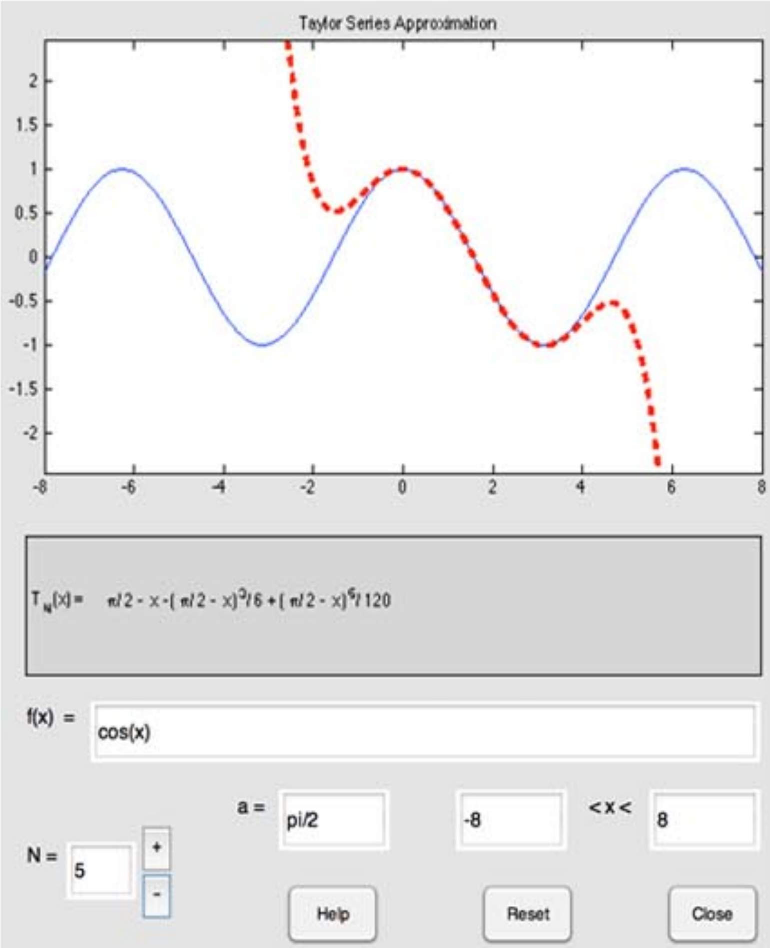
and the series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$ converges

to $\cos x \quad \forall x \in \mathbb{R}$.

ie, $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!} = \cos x.$$

Again, this means $\underbrace{\text{the sequence of partial sums}}_{T_N(x)}$ taylor polynomials

of the series converges to $\cos x$.


Taylor Series Approximation
$T_N(x) = \pi/2 - x - (\pi/2 - x)^3/6 + (\pi/2 - x)^5/120$
f(x) =
cos(x)
a =
pi/2
-8
< x <
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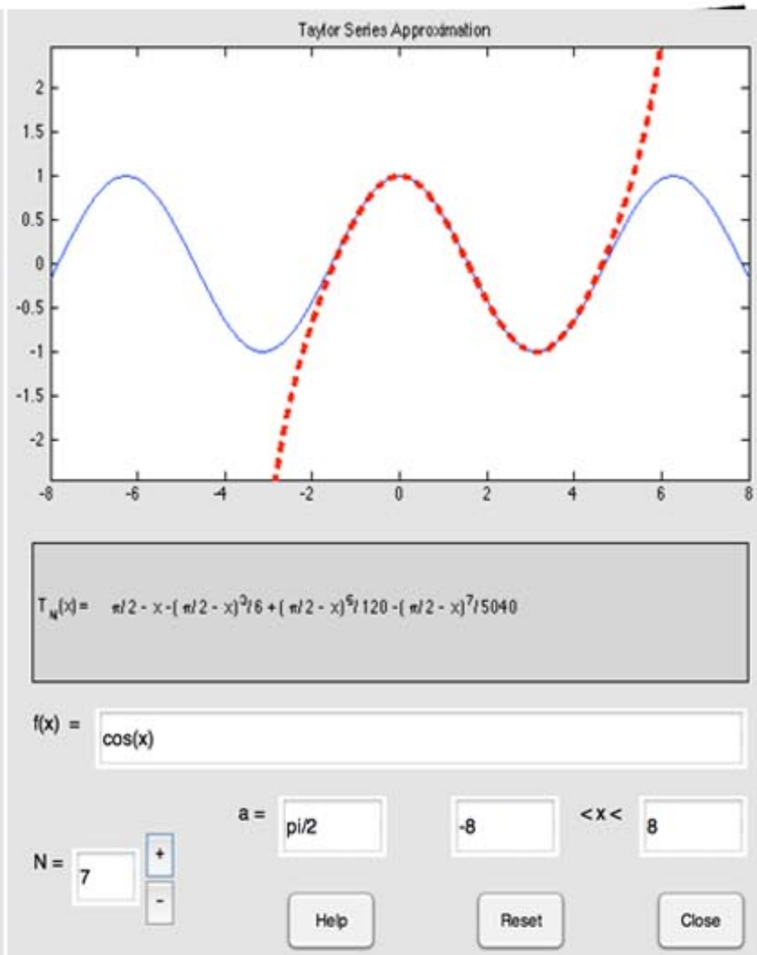
Taylor Series Approximation
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f(x) =
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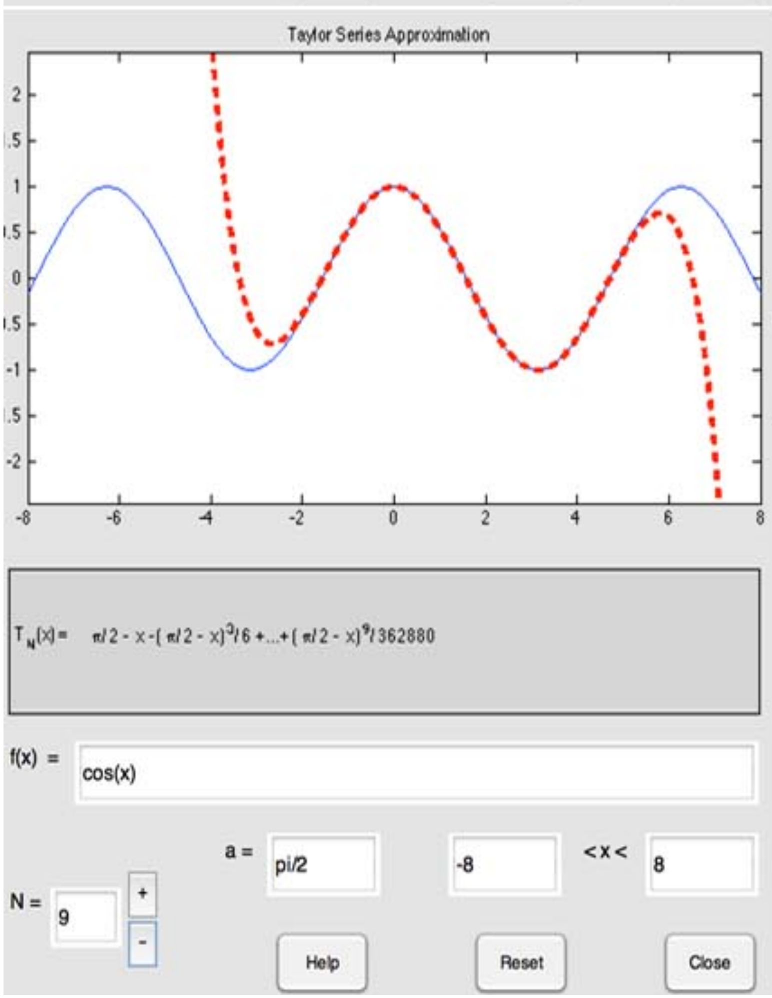
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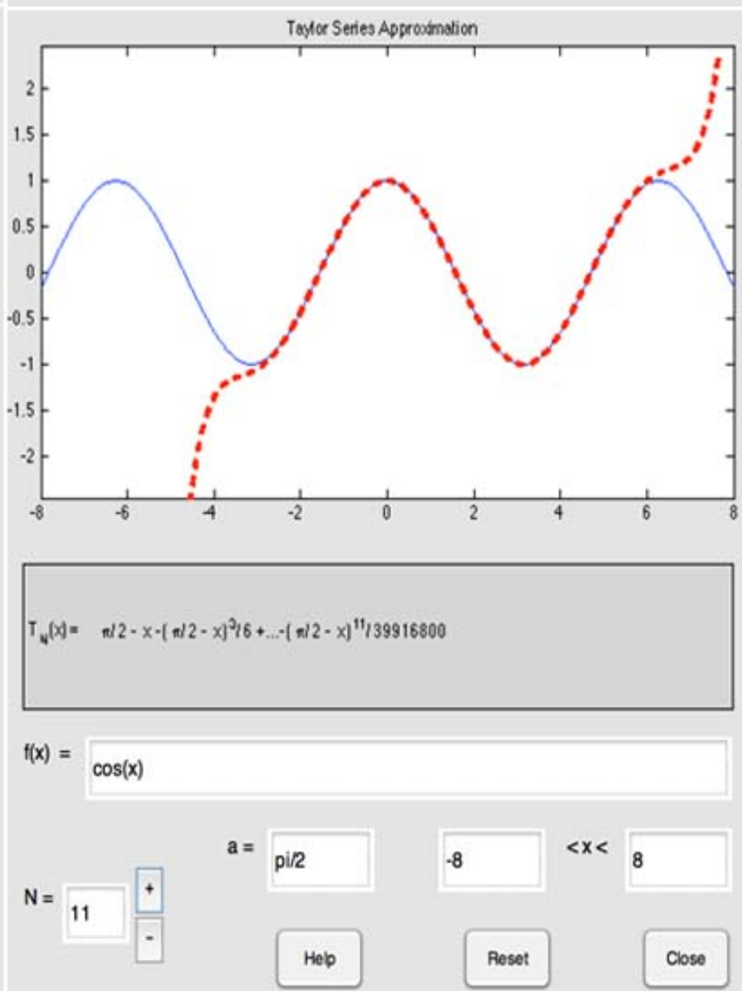
Taylor Series Approximation
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f(x) =
cos(x)
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Common Taylor and Maclaurin Series				
$\frac{1}{1-x}$	$=$	$1+x+x^2+x^3+\ldots$	$= \sum_{n=0}^{\infty} x^n$	$\lvert x\rvert<1$
e^x	$=$	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$	$= \sum_{n=0}^{\infty}\frac{x^n}{n!}$	all real x
$\sin x$	$=$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$	$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$	all real x
$\cos x$	$=$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$	$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$	all real x
$\ln(1+x)$	$=$	$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$	$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1} x^n}{n}$	$-1<x\le 1$
$\arctan x$	$=$	$x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$	$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$	$\lvert x\rvert\le 1$