Math 20300
Calculus III

Lesson 30
Series

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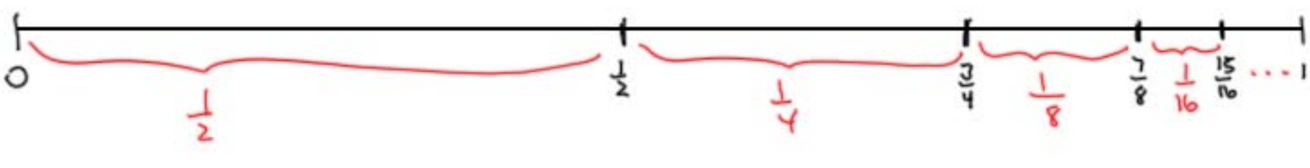
Series

A series is the summation of the terms in a sequence, ie $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

Could the sum of infinitely many numbers ever be finite?

Yes, we saw this in calc II with improper integrals:



$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

$$a_n = \frac{1}{2^n} \qquad \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Partial Sums

For a series $\sum_{i=1}^{\infty} a_i$, we consider

the partial sum $S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$

If the sequence $\{S_n\}$ converges to S $\left(\begin{matrix}\text{real \#,} \\ \text{not } \pm\infty\end{matrix}\right)$

then we say the series $\sum_{i=1}^{\infty} a_i$ converges

to S and we write $\sum_{i=1}^{\infty} a_i = S$

"the sum of the series."

If $\{S_n\}$ diverges, we say $\sum_{i=1}^{\infty} a_i$ diverges also.

For the series above $\sum_{k=1}^{\infty} \frac{1}{2^k}$, notice $S_1 = \sum_{k=1}^{1} \frac{1}{2^k} = \frac{1}{2}$

$$S_2 = \sum_{k=1}^{2} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \sum_{k=1}^{3} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$S_n = \sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1$$

$$\therefore \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

Theorem: If the series $\sum_{k=1}^{\infty} a_k$ is convergent,

then $\lim_{k\to\infty} a_k = 0$.

Proof: let $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_{n-1} + a_n$

then $S_{n-1} = \sum_{k=1}^{n-1} a_k = a_1 + a_2 + \cdots + a_{n-1}$

then $\lim_{n\to\infty} S_n = S$ and $\lim_{n\to\infty} S_{n-1} = S$

since the series converges.

Notice $S_n - S_{n-1} = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = a_n$

so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$

$= S - S = 0.$

$\therefore \lim_{n\to\infty} a_n = 0.$

A test for divergence:

If $\lim_{k\to\infty} a_k$ DNE or $\neq 0$, $\sum_{k=1}^{\infty} a_k$ diverges.

Ex. $\sum_{k=1}^{\infty} k = 1 + 2 + 3 + \cdots$ $\quad \lim_{k\to\infty} k = \infty \neq 0$

$\therefore \sum_{k=1}^{\infty} k$ diverges.

But consider $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ harmonic series

$$a_k = \frac{1}{k} \quad \text{and} \quad \lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{1}{k} = 0$$

HOWEVER, $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge. (it diverges).

Proof: Consider $S_2, S_4, S_8, \ldots$ (partial sums)

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2}$$

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$
$$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + \frac{3}{2}$$

$$S_{16} = \cdots > 1 + \frac{4}{2}$$

$S_{2^n} > 1 + \frac{n}{2}$ $\quad \therefore S_{2^n} \to \infty$ and $\{S_n\}$ diverges.

Series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots \qquad (a \neq 0)$$

are called <u>geometric series</u>.

Ex. $$2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots = \sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{3}\right)^{n-1} = \sum_{k=0}^{\infty} 2\left(\frac{1}{3}\right)^k$$

$k = n-1$

Ex. $$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^{n-1} = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k$$

The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ is

convergent for $|r| < 1$ and

the sum is $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

if $|r| \geq 1$, the series is divergent.

Notice $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{k=0}^{\infty} ar^{k}$ another way we'll see geometric series.

let $k = n-1$

Ex. $\sum_{n=1}^{\infty} 2\cdot\left(\frac{1}{3}\right)^{n-1} = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = \frac{2}{1}\cdot\frac{3}{2} = 3.$

$r = \frac{1}{3}$

$\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^{n-1} = \frac{1}{1+\frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}.$

$r = -\frac{1}{2}$

$\sum_{n=1}^{\infty} (5)^{n-1}$ diverges.

$r = 5$

Properties of convergent series:

1) $\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n$

2) $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

For the following examples, determine if the series converges or diverges. If it converges, find the sum.

Ex. $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{2^k}-\frac{1}{3^k}\right) = \sum_{k=0}^{\infty}\frac{1}{2^k} - \sum_{k=0}^{\infty}\frac{1}{3^k}$

$$= \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k - \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k = \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{3}}$$

$$= \frac{1}{\frac{1}{2}} - \frac{1}{\frac{2}{3}} = 2-\frac{3}{2} = \frac{1}{2}.$$

converges to $\frac{1}{2}$.

Ex. $\displaystyle\sum_{k=1}^{\infty}\frac{4k}{k+2}$ notice $\displaystyle\lim_{k\to\infty}\frac{4k}{k+2} = 4 \neq 0$

$\therefore$ The series diverges

Ex. $\displaystyle\sum_{k=0}^{\infty} \frac{2}{k+1} = \sum_{n=1}^{\infty} \frac{2}{n} = 2 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{harmonic series}}$ diverges

$n = k+1$

Ex. $\displaystyle\sum_{k=1}^{\infty} \frac{9}{k(k+3)} = \sum_{k=1}^{\infty} \left(\frac{3}{k} - \frac{3}{k+3}\right) =$

$$\frac{A}{k} + \frac{B}{k+3} = \frac{A(k+3) + Bk}{k(k+3)} \quad \therefore \; A(k+3) + Bk = 9 \;\; \forall k$$

$k = 0, \; A = 3$

$k = -3, \; B = -3$

$$= \underset{k=1}{\left(3 - \cancel{\tfrac{3}{4}}\right)} + \underset{k=2}{\left(\tfrac{3}{2} - \cancel{\tfrac{3}{5}}\right)} + \underset{k=3}{\left(\tfrac{3}{3} - \cancel{\tfrac{3}{6}}\right)} + \underset{k=4}{\left(\cancel{\tfrac{3}{4}} - \tfrac{3}{7}_{k=7}\right)}$$

$$+ \underset{k=5}{\left(\cancel{\tfrac{3}{5}} - \tfrac{3}{8}_{k=8}\right)} + \underset{k=6}{\left(\cancel{\tfrac{3}{6}} - \tfrac{3}{9}_{k=9}\right)} + \cdots$$

$$= 3 + \frac{3}{2} + \frac{3}{3} = 5\tfrac{1}{2} = \frac{11}{2}.$$

More formally, we can look at the sequence of partial sums $\{S_N\}$:

$$S_3 = 3 - \frac{3}{4} + \frac{3}{2} - \frac{3}{5} + \frac{3}{3} - \frac{3}{6}$$

$$S_4 = 3 - \cancel{\tfrac{3}{4}} + \tfrac{3}{2} - \tfrac{3}{5} + \tfrac{3}{3} - \tfrac{3}{6} + \cancel{\tfrac{3}{4}} - \tfrac{3}{7}$$

$$S_5 = 3 - \cancel{\tfrac{3}{4}} + \tfrac{3}{2} - \cancel{\tfrac{3}{5}} + \tfrac{3}{3} - \tfrac{3}{6} + \cancel{\tfrac{3}{4}} - \tfrac{3}{7} + \cancel{\tfrac{3}{5}} - \tfrac{3}{8}$$

$$S_N = 3 + \frac{3}{2} + \frac{3}{3} - \frac{3}{N+1} - \frac{3}{N+2} - \frac{3}{N+3} \quad \text{for } N \geq 3$$

and $\lim\limits_{N\to\infty} S_N = 3 + \frac{3}{2} + \frac{3}{3} = \frac{11}{2}$.

Ex. $\displaystyle\sum_{k=3}^{\infty} (-1)^k \frac{3}{2^k} = \sum_{k=3}^{\infty} 3\left(-\frac{1}{2}\right)^k$ geometric

$r = \frac{-1}{2}$ $|r| < 1$

converges

but $\displaystyle\underbrace{\sum_{k=0}^{\infty} 3\left(-\frac{1}{2}\right)^k} = \frac{3}{1+\frac{1}{2}} = \frac{3}{\frac{3}{2}} = 3 \cdot \frac{2}{3} = 2.$

$$\underset{k=0}{3} + \underset{k=1}{-\frac{3}{2}} + \underset{k=2}{\frac{3}{4}} + \sum_{k=3}^{\infty} 3\left(-\frac{1}{2}\right)^k = 2$$

then $\displaystyle\sum_{k=3}^{\infty} 3\left(-\frac{1}{2}\right)^k = 2 - 3 + \frac{3}{2} - \frac{3}{4} = -\frac{1}{4}.$

So far...

tools for convergence/divergence of $\sum_{k=1}^{\infty} a_k$:

- test for divergence $\lim_{k \to \infty} a_k \neq 0$

- geometric with $|r| < 1$ converges to $\frac{a}{1-r}$

  $|r| \geq 1$ diverges

- examine partial sums