Math 20300
Calculus III

Lesson 29
Sequences

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Sequences

A sequence is an ordered list of numbers $\{a_1, a_2, a_3, \ldots a_n \ldots\}$ with an infinite number of terms.

Ex. $$\{\underset{\substack{\uparrow\\ n=1\\ 2(0)}}{0}, \underset{\substack{\uparrow\\ n=2\\ 2(1)}}{2}, \underset{\substack{\uparrow\\ n=3\\ 2(2)}}{4}, \underset{\substack{\nwarrow\\ n=4\\ 2(3)}}{6}, \ldots\} = \{2(n-1)\} = \{2(n-1)\}_{n=1}^{\infty}$$

$$2(n-1)$$

Ex. $$\left\{\underset{\substack{\uparrow\\ n=1}}{\frac{1}{3}}, \underset{\substack{\uparrow\\ n=2}}{\frac{1}{9}}, \underset{\substack{\uparrow\\ n=3}}{\frac{1}{27}}, \ldots\right\} = \left\{\frac{1}{3^n}\right\}$$

Ex. $$\{\underset{\substack{\uparrow\\ n=1}}{1}, \underset{\substack{\uparrow\\ n=2}}{-1}, \underset{\substack{\uparrow\\ n=3}}{1}, \underset{\substack{\uparrow\\ n=4}}{-1}, \ldots\} = \left\{(-1)^{n+1}\right\}$$

Notation $\{a_1, a_2, a_3, \ldots\}$ $\{a_n\}$ $\{a_n\}_{n=1}^{\infty}$

Ex. Find a formula for the general term a_n of the sequence: $\left\{ \underset{\overset{\uparrow}{n=1}}{-\frac{1}{2}},\ \underset{\overset{\uparrow}{n=2}}{\frac{1}{3}},\ \underset{\overset{\uparrow}{n=3}}{-\frac{1}{4}},\ \dots \right\}$

$$a_n = \frac{(-1)^n}{n+1}$$

Ex. Find a formula for the general term a_n of the sequence: $\left\{ \underset{\overset{\uparrow}{n=1}}{1},\ \underset{\overset{\uparrow}{n=2}}{\frac{5}{2}},\ \underset{\overset{\uparrow}{n=3}}{\frac{25}{6}},\ \underset{\overset{\uparrow}{n=4}}{\frac{125}{24}},\ \dots \right\}$

$$a_n = \frac{5^{n-1}}{n!}$$

$1! = 1$

$2! = 2(1) = 2$

$3! = 3(2)(1) = 6$

$4! = 4(3)(2)(1) = 24$

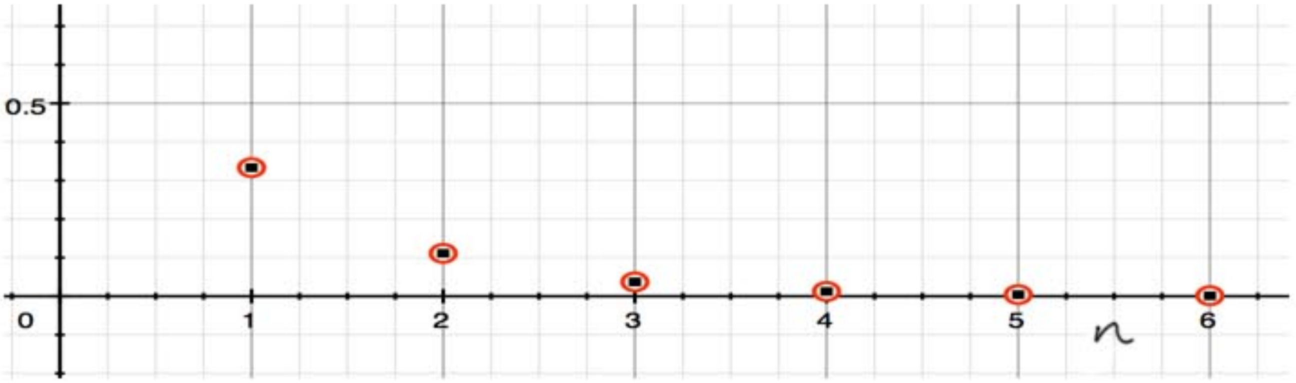
It is often convenient to consider the function $f(x)$ for which $f(n) = a_n$ where $n \in \{1, 2, 3, \dots\}$ or $n \in \{0, 1, 2, \dots\}$ as per the sequence definition.

Ex above: $\left\{ \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots \right\} = \left\{ \frac{1}{3^n} \right\}$

$$f(n) = \frac{1}{3^n} \qquad f(x) = \frac{1}{3^x}$$

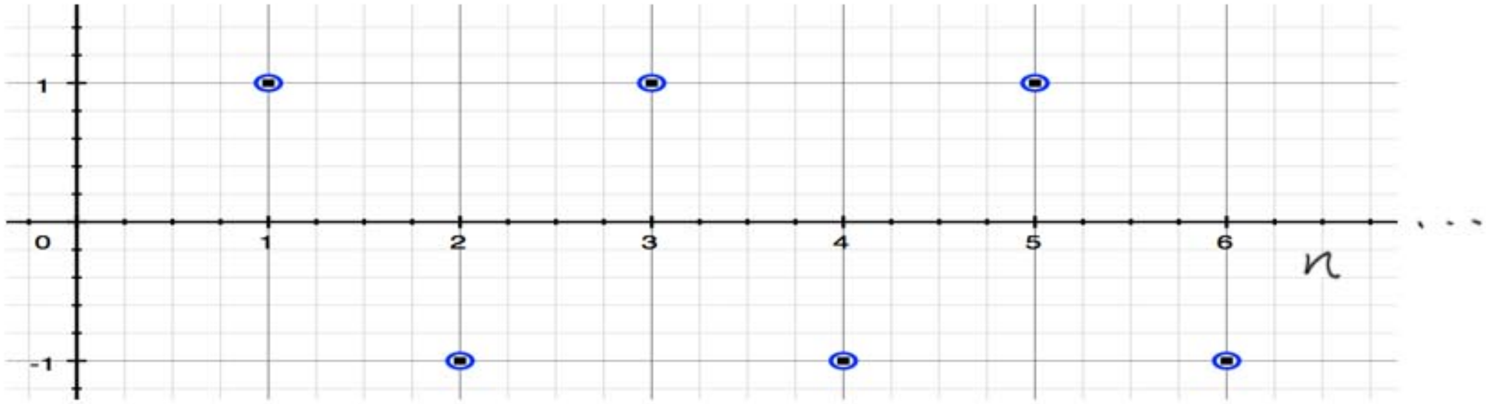
* Keep in mind factorial only makes sense for integers $n \geq 0$ $\quad (0! = 1)$

Graphs of Sequences:



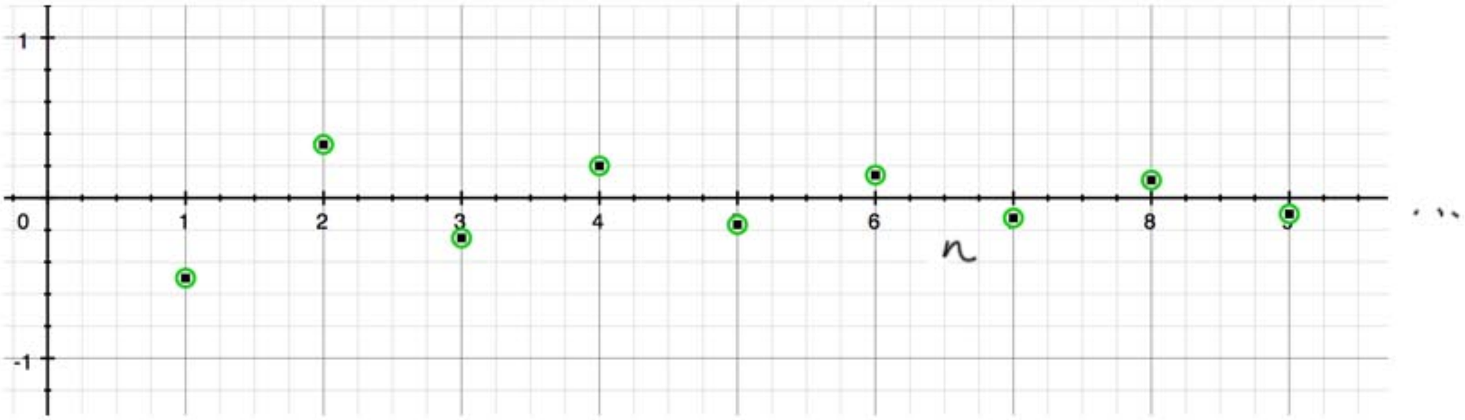
. . .

$\left\{ \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots \right\}$ $\quad \left\{ \frac{1}{3^n} \right\}$ looks like $\frac{1}{3^n} \to 0$ as $n \to \infty$



$\left\{ 1, -1, 1, -1, \ldots \right\} \quad \left\{ (-1)^{n+1} \right\}$

Keeps oscillating, values do not approach a limit



$\left\{ -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots \right\} \quad \left\{ \frac{(-1)^n}{n+1} \right\}$

looks like $\frac{(-1)^n}{n+1} \to 0$ as $n \to \infty$.

The Limit of a sequence:

A sequence $\{a_n\}$ has the limit L and we write $\lim\limits_{n\to\infty} a_n = L$ or $a_n \to L$

$$\text{if } \underbrace{\forall \varepsilon > 0}\ \underbrace{\exists N^{\swarrow \text{integer}} \ni n > N} \Rightarrow \underbrace{|a_n - L| < \varepsilon}.$$

however close you want to get to the limit L

we can find an N such that all the terms after a_N are as close as you wanted to L.

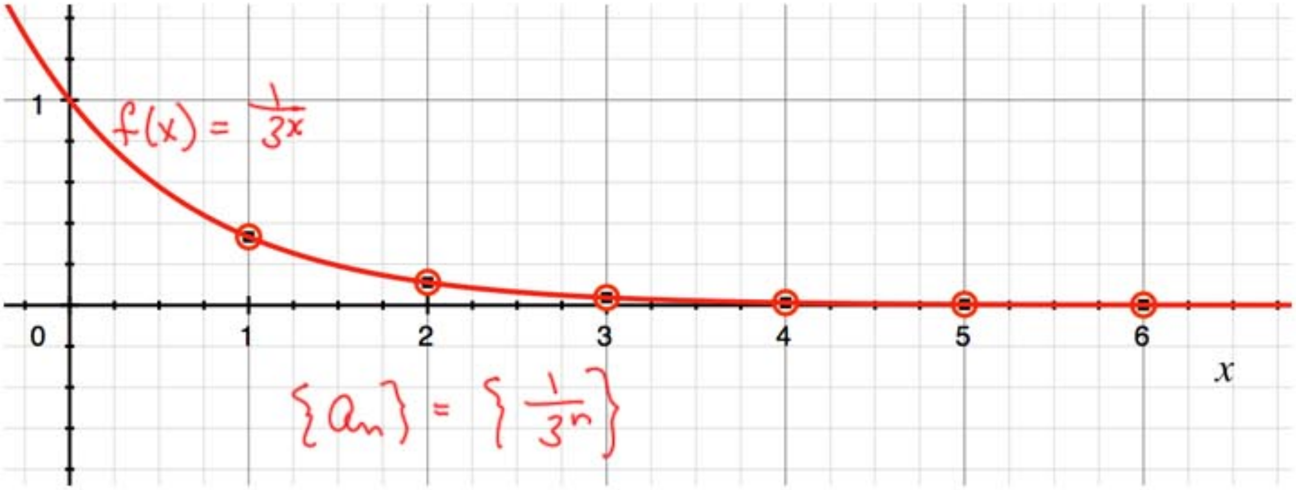
If $\lim\limits_{n\to\infty} a_n$ exists (as a real number, not $\pm\infty$), we say $\{a_n\}$ converges. Otherwise $\{a_n\}$ diverges.

We say $\lim\limits_{n\to\infty} a_n = \infty$ if $\underbrace{\forall \text{ positive integers } M},\ \underbrace{\exists N (\text{integer}) \ni n > N} \Rightarrow \underbrace{a_n > M}$.

however big you want the sequence terms to get,

we can find an N such that all the terms after a_N are as big as you wanted.

Theorem: If $\lim\limits_{x\to\infty} f(x) = L$ and $a_n = f(n)$ for positive integers n, then $\lim\limits_{n\to\infty} a_n = L$.



here we know $\lim\limits_{x\to\infty} \frac{1}{3^x} = 0$, $\therefore \lim\limits_{n\to\infty} \frac{1}{3^n} = 0$.

So $\left\{\frac{1}{3^n}\right\}$ converges to 0.

Tools for finding The limit of a sequence:

① If $\lim\limits_{x\to\infty} f(x) = L$ then $\lim\limits_{n\to\infty} f(n) = L$ (as above)

Note: this can include using L'Hospital's Rule to find $\lim\limits_{x\to\infty} f(x)$

② Properties of limits:

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant,

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n-b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c a_n = c\lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\cdot\left(\lim_{n\to\infty} b_n\right)$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{provided } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty}(a_n)^p = \left(\lim_{n\to\infty} a_n\right)^p \text{ if } p>0 \text{ and } a_n>0.$$

③ The Squeeze Theorem: If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

④ Theorem: If $\lim\limits_{n\to\infty} |a_n| = 0$, Then $\lim\limits_{n\to\infty} a_n = 0$.

⑤ The sequence $\{r^n\}$ is convergent if

$-1 < r \leq 1$ and divergent for all other r,

and $$\lim_{n\to\infty} r^n = \begin{cases} 0 & -1 < r < 1 \\ 1 & r = 1 \end{cases}.$$

Examples: Find $\lim\limits_{n\to\infty} a_n$, if it exists.

a) $\left\{\dfrac{(-1)^n n}{e^n}\right\}$ first lets get rid of The $(-1)^n$

$(-1)^x$ does not make sense

so we can examine $f(x) = \dfrac{x}{e^x}$

$\lim\limits_{x\to\infty} \dfrac{x}{e^x}$ $\dfrac{\infty}{\infty}$ use L'Hospital's Rule

$= \lim\limits_{x\to\infty} \dfrac{1}{e^x} = 0$. so $\dfrac{n}{e^n} \to 0$ as $n \to \infty$.

We know from ④ above That if $|a_n| \to 0$,

so does a_n. and

$$\left|\frac{(-1)^n n}{e^n}\right| = \frac{n}{e^n} \to 0, \quad \therefore \quad \frac{(-1)^n n}{e^n} \to 0.$$

This sequence converges to 0.

b) $\left\{ \frac{2(n+1)!}{n!} \right\}$ $\quad \lim_{n\to\infty} \frac{2(n+1)!}{n!} =$

$$= \lim_{n\to\infty} \frac{2(n+1)\cancel{n!}}{\cancel{n!}} =$$

$$= \lim_{n\to\infty} 2(n+1) = \infty.$$

This sequence diverges.

c) $\left\{4^{-n}\right\} = \left\{\frac{1}{4^n}\right\} = \left\{\left(\frac{1}{4}\right)^n\right\} = \left\{r^n\right\}$

Since $r = \frac{1}{4}$, $\lim_{n\to\infty} r^n = 0$

$$\therefore \lim_{n\to\infty} 4^{-n} = 0.$$

This sequence converges to 0.

d) $\left\{ \frac{\sin(n)}{n} \right\}$ we know $-1 \leq \sin(n) \leq 1$

and $n > 0 \ \forall n$, so $-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$

we know $\lim_{n\to\infty} -\frac{1}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ by The Squeeze Thm, $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$.

This sequence converges to 0.

Increasing, Decreasing, Monotonic, and Bounded Sequences:

$\{a_n\}$ is increasing if $a_n < a_{n+1} \ \forall n \geq 1$

$\{a_n\}$ is decreasing if $a_n > a_{n+1} \ \forall n \geq 1$

$\{a_n\}$ is monotonic if it is increasing or decreasing

$\{a_n\}$ is bounded above if $\exists M \ni$

$a_n \leq M \quad \forall n \geq 1$

$\{a_n\}$ is bounded below if $\exists m \in$

$m \leq a_n \quad \forall n \leq 1$.

$\{a_n\}$ is a bounded sequence if it is bounded above and below.

Monotonic Sequence Theorem: if $\{a_n\}$ is bounded and monotonic, it must be convergent.

Ex. $\left\{\frac{1}{n!}\right\}$ we know $\frac{1}{n!} > \frac{1}{(n+1)!} \quad \forall n \geq 1$

$\therefore \left\{\frac{1}{n!}\right\}$ is monotonically decreasing.

we also know $0 < \frac{1}{n!} \leq 1$

so $\left\{\frac{1}{n!}\right\}$ is bounded.

$\therefore \left\{\frac{1}{n!}\right\}$ converges. (converges to 0).