Math 20300
Calculus III

Lesson 27
Cylindrical Coordinates

Dr. A. Marchese, The City College of New York

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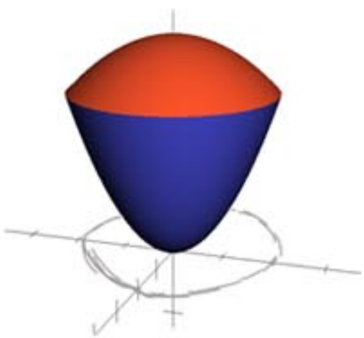
Cylindrical Coordinates

In lesson 26, we saw triple integrals in

r, θ, z instead of x, y, z :

Ex. Find the volume of the solid between

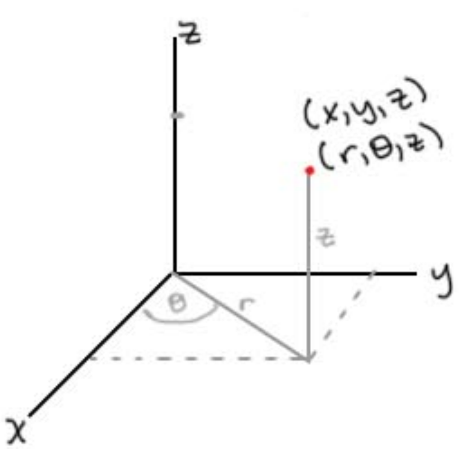
$$z = 16 - x^2 - y^2 \quad \text{and} \quad z = 3x^2 + 3y^2$$



$$\int_0^{2\pi} \int_0^{2} \int_{3r^2}^{16-r^2} 1 \, dz \; r \, dr \, d\theta$$

Using (r, θ, z) to describe points in $\mathbb{R}^3$

is the <u>Cylindrical Coordinate System</u>:



polar coordinates in $\mathbb{R}^2$

with the z-coordinate (height)

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

$x^2 + y^2 = r^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$

Ex. Convert to rectangular coordinates:

a) $(r, \theta, z) = (1, \frac{5\pi}{4}, 3)$

$x = r\cos\theta = 1\cos(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$

$y = r\sin\theta = 1\sin(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$

$(x, y, z) = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 3)$

b) $(r, \theta, z) = (2, \pi, -1)$

$x = r\cos\theta = 2\cos\pi = -2$

$y = r\sin\theta = 2\sin\pi = 0$

$(x, y, z) = (-2, 0, -1)$

Ex. Convert to cylindrical coordinates:

a) $(x, y, z) = (-\sqrt{3}, -1, -2)$

$r^2 = x^2 + y^2 = (-\sqrt{3})^2 + (-1)^2 = 4$

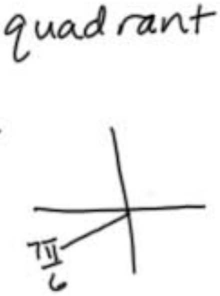
$r = \pm 2$ choose $r = 2$

$\tan\theta = \frac{y}{x} = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$

look at $\arctan(\frac{1}{\sqrt{3}})$ then check quadrant

$(x, y) = (-\sqrt{3}, -1)$ QIII

$\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$



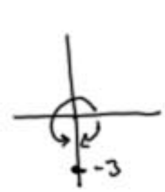
$\theta = \frac{7\pi}{6}$

b) $(x, y, z) = (0, -3, 1)$

$r^2 = x^2 + y^2 = 0^2 + (-3)^2 = 9$

$r = \pm 3$, choose $r = 3$.

$\tan\theta = \frac{y}{x} = \frac{-3}{0}$ DNE



$\theta = \frac{3\pi}{2}$ or $-\frac{\pi}{2}$

$y = r\sin\theta$

$-3 = 3\sin\theta$

$-1 = \sin\theta$

$\theta = \frac{3\pi}{2}$ or $-\frac{\pi}{2}$

$(r,\theta,z) = \left(2, \frac{7\pi}{6}, -2\right)$ $\qquad\qquad$ $(r,\theta,z) = \left(3, \frac{3\pi}{2}, 1\right)$

$\left(2, -\frac{5\pi}{6}, -2\right)$ $\qquad\qquad$ $\left(3, -\frac{\pi}{2}, 1\right)$

Surfaces in Cylindrical Coordinates:

Ex. Write the equations in cylindrical coordinates:

a) $z = \underbrace{x^2+y^2}_{r^2} - 2xy$

$z = r^2 - 2(r\cos\theta)(r\sin\theta)$

$z = r^2 - 2r^2\cos\theta\sin\theta$

$z = r^2(1 - 2\cos\theta\sin\theta)$

b) $\underbrace{x^2+y^2}_{r^2} + z^2 = 16$

$r^2 + z^2 = 16$

Ex. Convert the equations to rectangular coordinates:

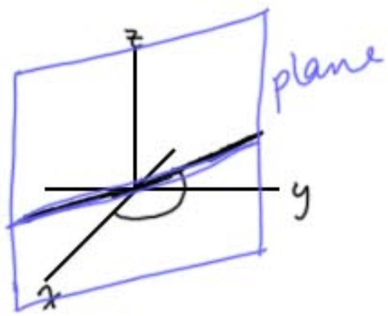
a) $r = 2$ we know $x^2 + y^2 = r^2$

$r^2 = 4$ then $x^2 + y^2 = 4$ cylinder.

b) $\theta = \frac{3\pi}{4}$ we know $x = r\cos\theta$

$$= r\cos\tfrac{3\pi}{4} = r\left(-\tfrac{\sqrt{2}}{2}\right)$$

$$x = -\tfrac{\sqrt{2}}{2}\, r$$



$$y = r\sin\theta$$
$$= r\sin\left(\tfrac{3\pi}{4}\right) = r\left(\tfrac{\sqrt{2}}{2}\right)$$
$$= \tfrac{\sqrt{2}}{2}\, r$$

$x = -y$ or $y = -x$. plane.

(More)

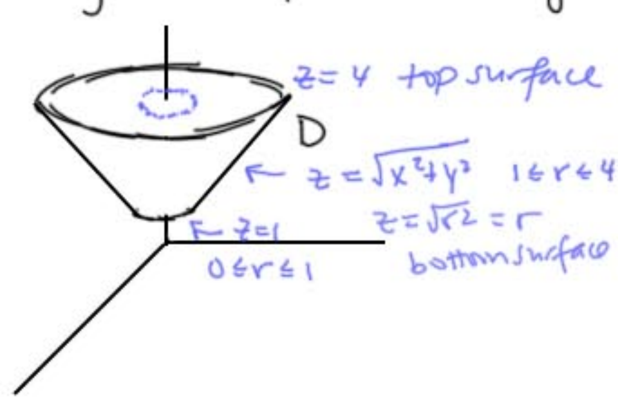
Triple Integrals in Cylindrical Coordinates:

Ex. Find the center of mass of the frustum of

the cone $z^2 = x^2 + y^2$

between $z = 1$ + $z = 4$

(assume $\rho(x,y,z) = 1$)



guess: $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} \approx 3$

$$dV = dz\, r\,dr\,d\theta$$
$$= r\,dz\,dr\,d\theta$$

$$\bar{z} = \frac{1}{\text{mass}(D)} \iiint_D z \; dV$$

$$\text{mass}(D) = \iiint_D dV = \int_0^{2\pi}\int_0^1\int_1^4 r\,dz\,dr\,d\theta \quad +$$

$$+ \int_0^{2\pi}\int_1^4\int_r^4 r\,dz\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^1 rz\Big|_1^4 \,dr\,d\theta \quad + \quad \int_0^{2\pi}\int_1^4 rz\Big|_r^4 \,dr\,d\theta \quad =$$

$$= \int_0^{2\pi}\int_0^1 3r \; dr\,d\theta \quad + \int_0^{2\pi}\int_1^4 (4r - r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{3r^2}{2}\Big|_0^1 \; d\theta \quad + \quad \int_0^{2\pi} \left(\frac{4r^2}{2} - \frac{r^3}{3}\right)\Big|_1^4 \; d\theta$$

$$= \int_0^{2\pi} \frac{3}{2}\, d\theta + \int_0^{2\pi} 9\, d\theta = \frac{21}{2}\theta\Big|_0^{2\pi} = 21\pi$$

$$= mass(D).$$

$$\bar{z} = \frac{1}{mass(D)} \iiint_D z \, dV$$

$$= \frac{1}{21\pi} \int_0^{2\pi}\int_0^1\int_1^4 zr \, dz\, dr\, d\theta + \frac{1}{21\pi}\int_0^{2\pi}\int_1^4\int_r^4 zr\, dz\, dr\, d\theta$$

$$= \frac{1}{21\pi}\int_0^{2\pi}\int_0^1 \frac{z^2}{2}r\Big|_1^4 dr\, d\theta + \frac{1}{21\pi}\int_0^{2\pi}\int_1^4 \frac{z^2}{2}r\Big|_r^4 dr\, d\theta$$

$$= \frac{1}{21\pi}\int_0^{2\pi}\int_0^1 \frac{15}{2} r\, dr\, d\theta + \frac{1}{21\pi}\int_0^{2\pi}\int_1^4 \left(8r - \frac{r^3}{2}\right) dr\, d\theta$$

$$= \frac{1}{21\pi}\int_0^{2\pi} \frac{15}{4}r^2\Big|_0^1 d\theta + \frac{1}{21\pi}\int_0^{2\pi}\left(\frac{8r^2}{2} - \frac{r^4}{8}\right)\Big|_1^4 d\theta$$

$$= \frac{1}{21\pi}\int_0^{2\pi}\left[\frac{15}{4} + 64 - \frac{256}{8} - 4 + \frac{1}{8}\right] d\theta$$

$$\frac{1}{21\pi}\int_0^{2\pi} \frac{255}{8}\, d\theta = \frac{1}{21\pi}\cdot\frac{255}{8}\,\theta\Big|_0^{2\pi} =$$

$$= \frac{1}{\cancel{21}_{7}\cancel{\pi}}\;\frac{\cancel{255}^{85}}{\cancel{8}_{4}}\cdot\cancel{2\pi} = \frac{85}{28} \approx 3.0357.$$

Check $\bar{x}$ and $\bar{y}$ as an exercise.

Ex. Evaluate by changing to cylindrical coordinates.

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{2-x^2-y^2} \sqrt{x^2+y^2}\; dz\, dy\, dx$$

1) change the inside function from $(x,y) \to (r,\theta)$

2) change the bounds on z from $(x,y) \to (r,\theta)$

3) use the bounds on y + x to draw the region in the x-y plane.

4) redescribe the region in (r,θ) and find the bounds for r + θ

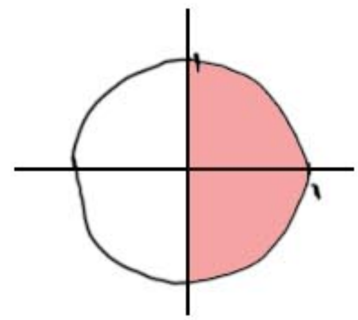
5) $dy\,dx = r\,dr\,d\theta$ $\quad (dx\,dy = r\,dr\,d\theta)$

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{2-x^2-y^2} \sqrt{\underbrace{x^2+y^2}_{r^2}}\, dz\, dy\, dx$$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$0 \le x \le 1$

$$\int_{-\frac{\pi}{2}}^{\pi/2} \int_0^1 \int_0^{2-r^2} r^2\, dz\, dr\, d\theta$$



$0 \le r \le 1$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\int_{-\frac{\pi}{2}}^{\pi/2} \int_0^1 r^2 z \Big|_0^{2-r^2} dr\, d\theta = \int_{-\frac{\pi}{2}}^{\pi/2} \int_0^1 (2r^2 - r^4)\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \left(\frac{2r^3}{3} - \frac{r^5}{5}\right)\Big|_0^1 d\theta = \int_{-\frac{\pi}{2}}^{\pi/2} \left(\frac{2}{3} - \frac{1}{5}\right) d\theta =$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \frac{7}{15}\, d\theta = \frac{7}{15}\theta \Big|_{-\frac{\pi}{2}}^{\pi/2} = \frac{7\pi}{15}.$$