Math 20300
Calculus III

Lesson 24
Applications of the Double Integral

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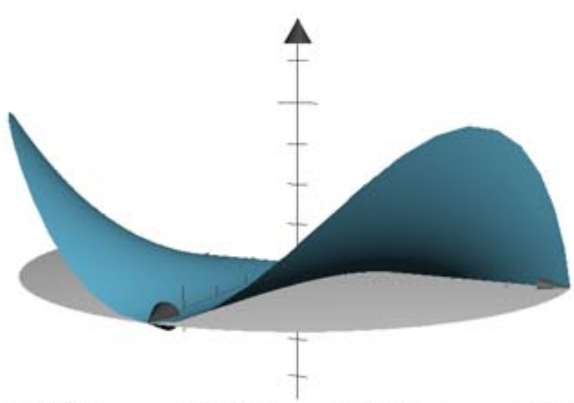
Bookmarks have been added to this video at the following times:

Applications of The Double Integral

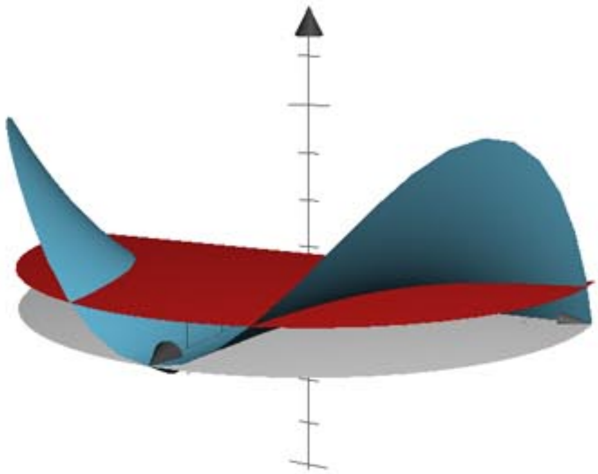
- volume - lessons 21, 22, 23
- area - lesson 23 volume under $f(x,y)=1$ same as area of domain
- average value of $f(x,y)$ over domain D
- total mass of a plate, given density
- center of mass of a plate, given density
- moment of inertia of a plate, given density
- surface area - lesson 25

Average value of $f(x,y)$ over domain D:

Ex.



$f(x,y) = xy^2$ over right half of unit disk



we're looking for

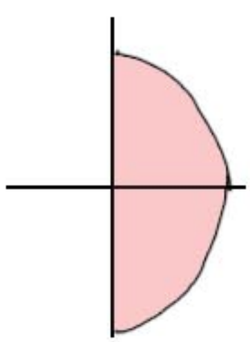
$z = \text{Avg}(f)$ such that

$$\text{Avg}(f) \cdot \underset{D}{\text{area of domain}} = \iint_D f(x,y)\,dA$$

So $$\text{Avg}(f) = \frac{1}{\text{area of } D} \iint_D f(x,y)\,dA$$

Ex above, avg value of $f(x,y) = xy^2$ over the right half of the unit disk.

D:



$$\text{area of } D = \tfrac{1}{2}(\text{area of circle})$$

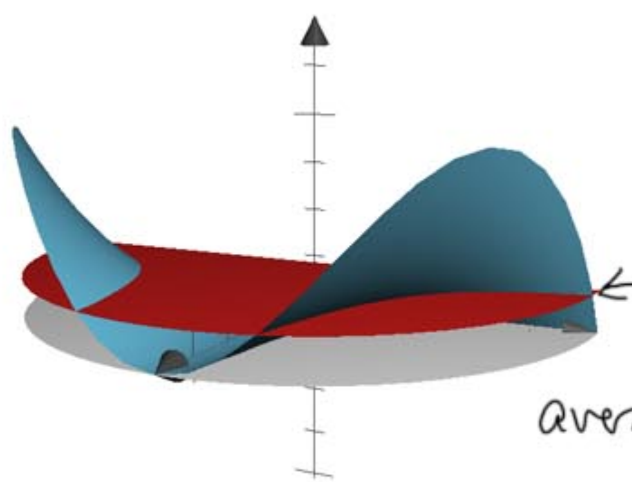
$$= \tfrac{1}{2}\pi(1)^2 = \frac{\pi}{2}$$

So $\quad \text{Avg}(f) = \dfrac{1}{\frac{\pi}{2}} \displaystyle\iint_D xy^2 \, dA =$

$$= \frac{2}{\pi} \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} xy^2 \, dx \, dy$$

$$\text{OR} = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\pi/2} \int_{0}^{1} r\cos\theta \cdot (r\sin\theta)^2 \, r \, dr \, d\theta$$

$$= \frac{4}{15\pi}$$ (compute as an exercise)



red plane height is $\frac{4}{15\pi}$ = average value of f over D.

For mass calculations:

Recall: $\text{density} = \dfrac{\text{mass}}{\text{volume}}$ for solids in $\mathbb{R}^3$

$\text{density} = \dfrac{\text{mass}}{\text{area}}$ for a "thin plate" in $\mathbb{R}^2$.

When density varies through the region in $\mathbb{R}^2$,

density $= \rho(x,y)$, and

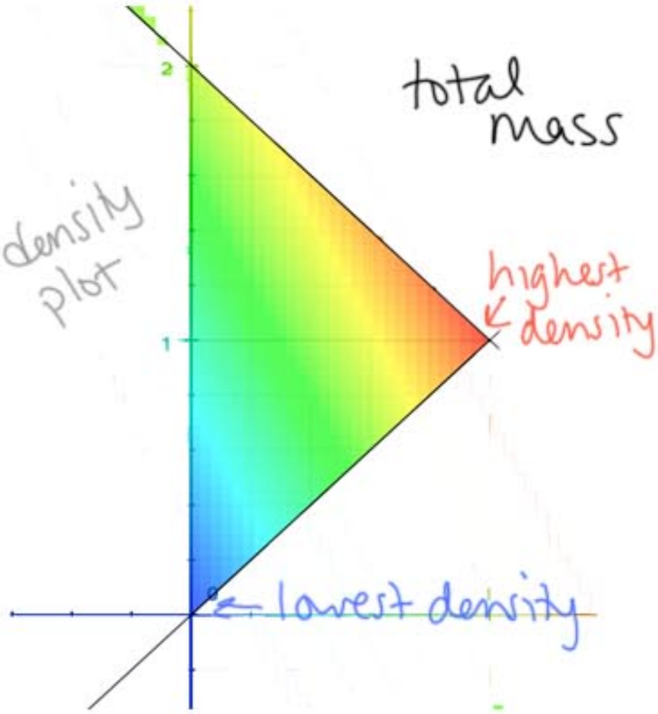
<u>Total Mass</u> over region D $= \displaystyle\iint_D \rho(x,y)\,dA$

Riemann sums break domain into rectangles, and on each we multiply density · area = mass

$$\sum_i \sum_j \rho(x_i^*, y_j^*) \cdot \Delta A_{ij}$$

Ex. Find the mass of a thin triangular plate bounded by the y axis and the lines

$y = x$ and $y = 2 - x$, w/ $\rho(x,y) = 6x + 3y + 3$.



$$\text{total mass} = \int_0^1 \int_x^{2-x} (6x + 3y + 3)\, dy\, dx$$

$$= \int_0^1 \left[6xy + 3\frac{y^2}{2} + 3y \right]_x^{2-x} dx$$

$$= \int_0^1 \left[\left(6x(2-x) + 3\frac{(2-x)^2}{2} + 3(2-x) \right) - \left(6x^2 + 3\frac{x^2}{2} + 3x \right) \right] dx$$

$$= \int_0^1 \left[12x - 6x^2 + \frac{3}{2}(4 - 4x + x^2) + 6 - 3x - 6x^2 - \frac{3}{2}x^2 - 3x \right] dx$$

$$= \int_0^1 \left[-12x^2 + 12 \right] dx = -\frac{12x^3}{3} + 12x \Big|_0^1 = -4x^3 + 12x \Big|_0^1$$

$$= (-4 + 12) - (0 + 0) = 8.$$

Center of Mass of a thin plate:

the (x,y) point on the plate at which it could be suspended and remain in static equilibrium.

(ie, point at which you could balance the plate on your finger tip)

for the x-coordinate, need

$$(\text{total mass})(x \text{ coord of center of mass}) = \iint_D x\,\rho(x,y)\,dA$$

for the y-coordinate, need

$$(\text{total mass})(y \text{ coord of center of mass}) = \iint_D y\,\rho(x,y)\,dA$$

so $$\bar{x} = \frac{1}{M}\iint_D x\,\rho(x,y)\,dA$$

$$M = \iint_D \rho(x,y)\,dA$$

$$\bar{y} = \frac{1}{M}\iint_D y\,\rho(x,y)\,dA$$

Ex using plate and density above, triangular region with $\rho(x,y) = 6x+3y+3$, we found total mass $M = 8$. Find the center of mass of the plate:

$$\bar{x} = \frac{1}{M}\iint_D x\,\rho(x,y)\,dA = \frac{1}{8}\int_0^1\int_x^{2-x} x(6x+3y+3)\,dy\,dx$$

$$= \frac{1}{8}\int_0^1\int_x^{2-x}(6x^2+3xy+3x)\,dy\,dx$$

$$= \frac{1}{8}\int_0^1\left[6x^2y+\frac{3xy^2}{2}+3xy\right]_x^{2-x}dx$$

$$= \frac{1}{8}\int_0^1\left[\left(6x^2(2-x)+\frac{3x(2-x)^2}{2}+3x(2-x)\right)-\left(6x^3+\frac{3x^3}{2}+3x^2\right)\right]dx$$

$$= \frac{1}{8}\int_0^1\Big[12x^2-6x^3+\tfrac{3}{2}x(4-4x+x^2)+6x-3x^2-6x^3-\tfrac{3}{2}x^3$$

$6x-6x^2+\frac{3}{2}x^3$

$$-3x^2\Big]\,dx$$

$$= \frac{1}{8}\int_0^1 (-12x^3 + 12x)\,dx = \frac{1}{8}\left(-\frac{12x^4}{4} + \frac{12x^2}{2}\right)\Bigg|_0^1$$

$$= \frac{1}{8}\left(-3x^4 + 6x^2\right)\Big|_0^1 = \frac{1}{8}(-3+6) = \frac{3}{8} = \bar{x}$$

$$\bar{y} = \frac{1}{M}\iint_D y\,\rho(x,y)\,dA = \frac{1}{8}\int_0^1\int_x^{2-x} y(6x+3y+3)\,dy\,dx$$

$$= \frac{1}{8}\int_0^1\int_x^{2-x} (6xy + 3y^2 + 3y)\,dy\,dx =$$

$$= \frac{1}{8}\int_0^1 \left(\frac{6xy^2}{2} + \frac{3y^3}{3} + \frac{3y^2}{2}\right)\Bigg|_x^{2-x} dx$$

$$= \frac{1}{8}\int_0^1 \left(3xy^2 + y^3 + \frac{3}{2}y^2\right)\Bigg|_x^{2-x} dx$$

$$= \frac{1}{8}\int_0^1 \left[\left(3x(2-x)^2 + (2-x)^3 + \frac{3}{2}(2-x)^2 - \left(3x^3 + x^3 + \frac{3}{2}x^2\right)\right] dx$$

$$= \frac{1}{8}\int_0^1 \left[(2-x)^2\left(3x+2-x+\frac{3}{2}\right) - 4x^3 - \frac{3}{2}x^2\right] dx$$

$$(4-4x+x^2)\left(2x+\frac{7}{2}\right)$$

$$= \frac{1}{8}\int_0^1 \left(8x + 14 - 8x^2 - 14x + 2x^3 + \frac{7}{2}x^2 - 4x^3 - \frac{3}{2}x^2\right) dx$$

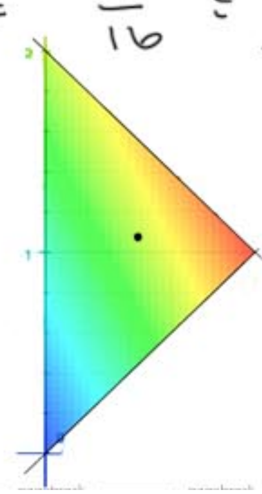
$$= \frac{1}{8}\int_0^1 \left(-2x^3 - 6x^2 - 6x + 14\right) dx$$

$$= \frac{1}{8}\left(-\frac{2}{4}x^4 - \frac{6x^3}{3} - \frac{6x^2}{2} + 14x\right)\Big|_0^1 =$$

$$= \frac{1}{8}\left(-\frac{x^4}{2} - 2x^3 - 3x^2 + 14x\right)\Big|_0^1$$

$$= \frac{1}{8}\left(-\frac{1}{2} - 2 - 3 + 14\right) = \frac{1}{8}\cdot\frac{17}{2} = \frac{17}{16} = \bar{y}$$

Center of mass: $\left(\frac{3}{8}, \frac{17}{16}\right)$



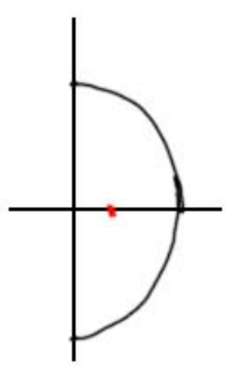
If $\rho(x,y)=1$, these become

$$M = \iint_D \rho(x,y)\,dA = \iint_D dA = \text{area of } D$$

$$\bar{x} = \frac{1}{M}\iint_D x\,\rho(x,y)\,dA = \frac{1}{\text{area of } D}\iint_D x\,dA = \text{avg value of } f(x,y)=x \text{ over } D$$

$$\bar{y} = \frac{1}{M}\iint_D y\,\rho(x,y)\,dA = \frac{1}{\text{area of } D}\iint_D y\,dA = \text{avg value of } f(x,y)=y \text{ over } D$$

Ex. Find the center of mass of the right half of the unit disk with constant density $=1$.



we expect $\bar{y}=0$ by symmetry,

but $$\bar{x} = \frac{1}{\frac{\pi}{2}}\iint_D x\,dA$$

$$= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\pi/2}\int_0^1 r\cos\theta\, r\,dr\,d\theta$$

$$= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\pi/2} \frac{r^3}{3}\cos\theta\Big|_0^1\,d\theta$$

$$= \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{3}\cos\theta \, d\theta = \frac{2}{3\pi} \sin\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{3\pi}(1+1)$$

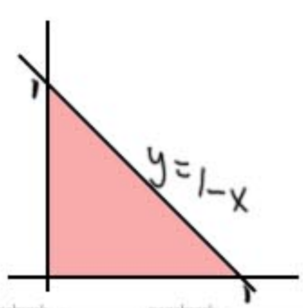
$$= \frac{4}{3\pi} \approx .42$$

Moment of Inertia: the tendency to resist angular acceleration

around y axis: $I_y = \iint_D x^2 \rho(x,y)\, dA$

around x axis: $I_x = \iint_D y^2 \rho(x,y)\, dA$

Ex. Find the moments of inertia of the triangular region bounded by $x=0$, $y=0$, $y=1-x$ with constant density $\rho(x,y)=\rho$.



$$I_y = \int_0^1\int_0^{1-x} \rho x^2 \, dy\, dx =$$

$$= \int_0^1 \rho x^2 y \Big|_0^{1-x} dx = \int_0^1 \rho x^2(1-x)\, dx = \rho \int_0^1 (x^2 - x^3)\, dx$$

$$= \rho \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1 = \rho \left(\frac{1}{3} - \frac{1}{4} \right) = \rho \left(\frac{1}{12} \right) = \frac{\rho}{12}.$$

$$I_x = \int_0^1 \int_0^{1-x} \rho y^2\, dy\, dx = \int_0^1 \rho \frac{y^3}{3} \Big|_0^{1-x} dx = \int_0^1 \rho \frac{(1-x)^3}{3}\, dx$$

$u = 1-x$
$du = -dx$

$$= -\rho \int_1^0 \frac{u^3}{3}\, du = \rho \int_0^1 \frac{u^3}{3}\, du = \rho \frac{u^4}{12} \Big|_0^1 = \frac{\rho}{12} = I_y$$ expected by symmetry