Math 20300
Calculus III

Lesson 23
Double Integrals In Polar Coordinates

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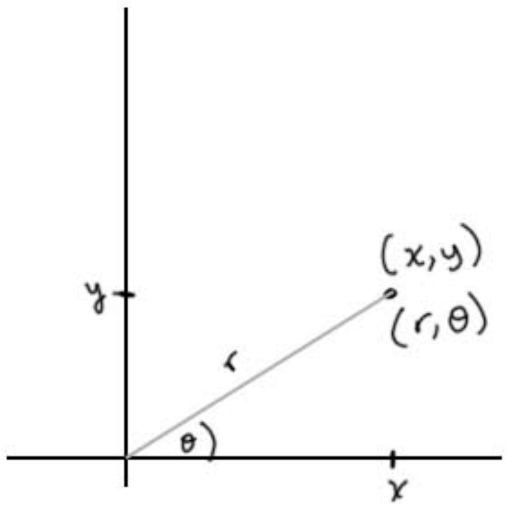
Double Integrals in Polar Coordinates

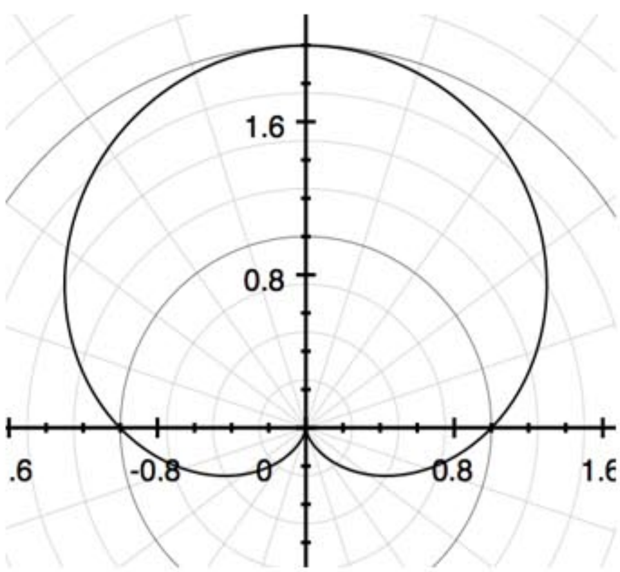
Recall

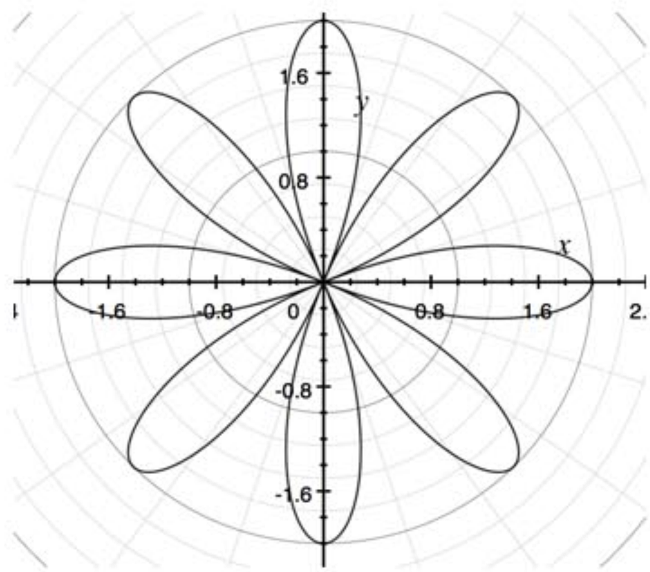


$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

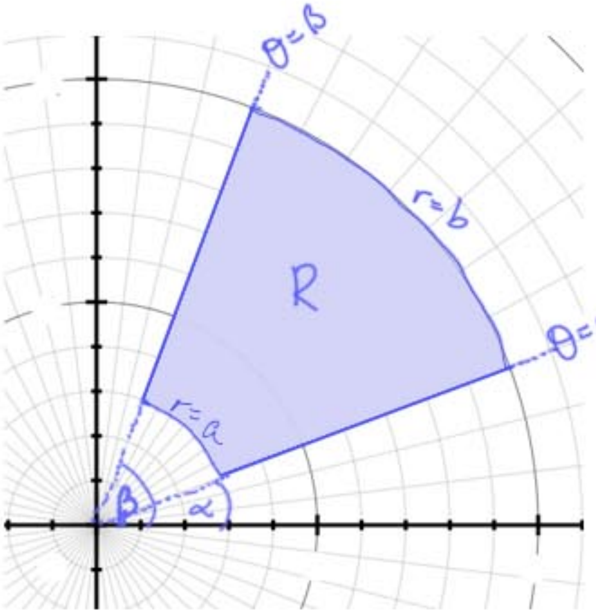


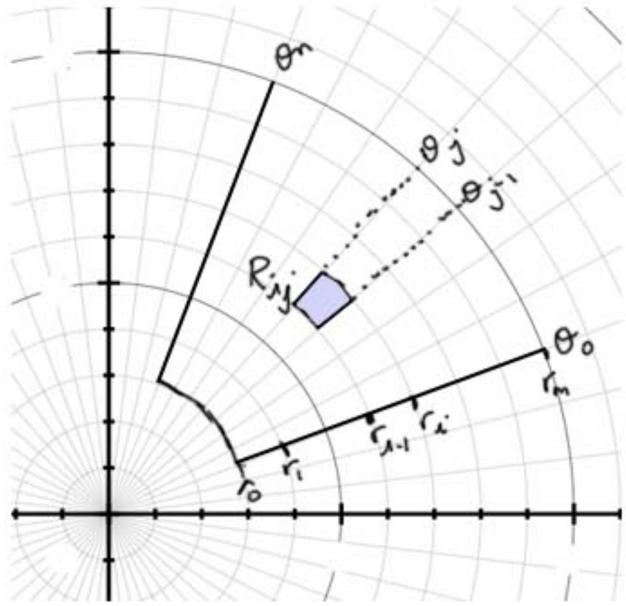


$r = 1 + \sin\theta$

$r = 2\cos(4\theta)$

$r = f(\theta)$



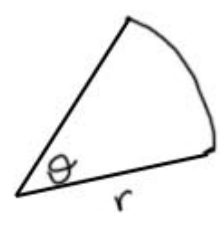


$R = \{(r,\theta) : a \le r \le b, \alpha \le \theta \le \beta\}$ need the area of R_{ij}

Polar rectangle

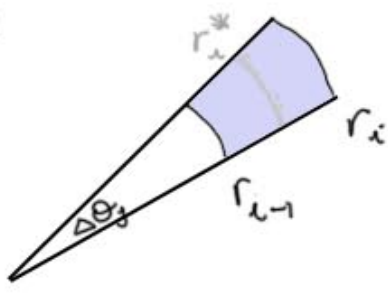
Recall: area of circle $A = \pi r^2$ notice $\theta = 2\pi$ for whole circle

area of sector of angle θ $A = \frac{\theta}{2} r^2$



for R_{ij} we are subtracting areas:

$\Delta\theta_j = \theta_j - \theta_{j-1}$



$$\frac{\Delta\theta_j}{2}(r_i)^2 - \frac{\Delta\theta_j}{2}(r_{i-1})^2$$

$$= \frac{\Delta\theta_j}{2}\left(r_i^2 - r_{i-1}^2\right)$$

$$= \frac{\Delta\theta_j}{2}(r_i + r_{i-1})\underbrace{(r_i - r_{i-1})}_{\Delta r_i}$$

$$= \frac{(r_i + r_{i-1})}{2}\Delta r_i \Delta\theta_j$$

$$= r_i^* \Delta r_i \Delta\theta_j$$

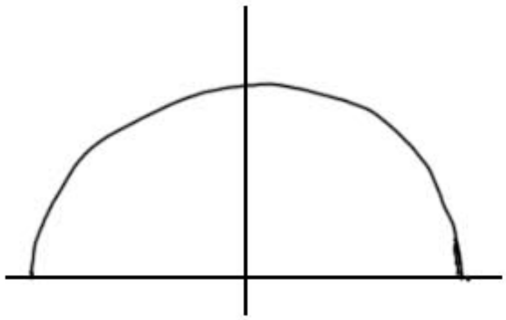
Then the Riemann sums of lesson 21:

$$\iint_R f(x,y)\,dA = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$

$$= \lim_{\substack{\max \Delta r_i \to 0 \\ \max \Delta \theta_j \to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\, r_i^* \Delta r_i \Delta\theta_j$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Ex. Find the double integral of $f(x,y) = x^2 + y$ over the top half of the unit disk.



in cartesian coordinates,

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2 + y)\,dy\,dx$$

in polar coordinates

$$\int_{0}^{\pi}\int_{0}^{1} \left((r\cos\theta)^2 + r\sin\theta\right) r\,dr\,d\theta$$

$$= \int_{0}^{\pi}\int_{0}^{1} \left(r^3\cos^2\theta + r^2\sin\theta\right) dr\,d\theta$$

$$= \int_{0}^{\pi} \left[\frac{r^4}{4}\cos^2\theta + \frac{r^3}{3}\sin\theta\right]_0^1 d\theta$$

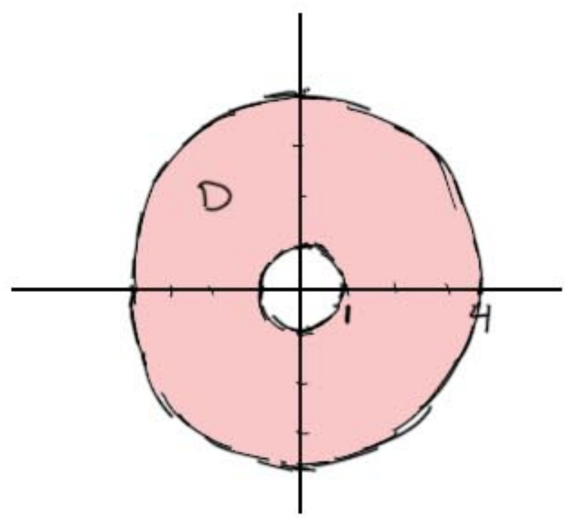
$$= \int_{0}^{\pi} \left(\frac{\cos^2\theta}{4} + \frac{\sin\theta}{3}\right) d\theta$$

$$= \int_0^{\pi} \left(\frac{1}{4}\cdot\frac{1}{2}(1+\cos 2\theta) + \frac{1}{3}\sin\theta\right) d\theta =$$

$$= \frac{1}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right) - \frac{1}{3}\cos\theta \Big|_0^{\pi}$$

$$= \frac{1}{8}(\pi) - \frac{1}{3}(-1) - \left(0 - \frac{1}{3}(1)\right) = \frac{\pi}{8} + \frac{2}{3}.$$

Ex. $\iint\limits_D (x^2+y^2)\, dA$



$f(x,y) = x^2 + y^2$

$f(r\cos\theta, r\sin\theta) = r^2\cos^2\theta + r^2\sin^2\theta = r^2(1) = r^2$

$$\iint\limits_D (x^2+y^2)\, dA = \int_0^{2\pi}\int_1^4 \underbrace{r^2 \cdot r}_{r^3}\, dr\, d\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_1^4 \, d\theta$$

$$= \int_0^{2\pi} \left(64 - \tfrac{1}{4}\right) d\theta = \int_0^{2\pi} \frac{255}{4}\, d\theta = \frac{255}{4}\theta \Big|_0^{2\pi} = \frac{255\pi}{2}.$$

If we integrate the function $f(x,y) = 1$

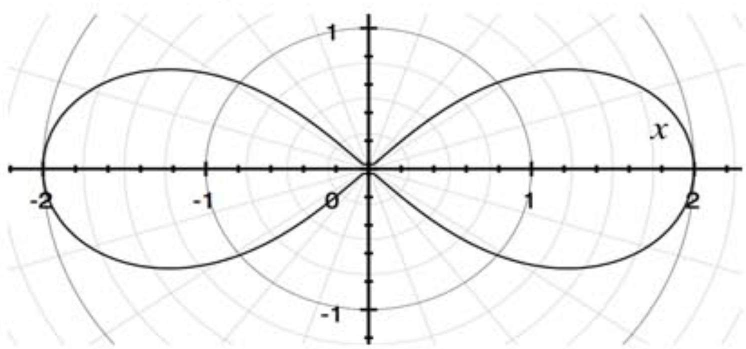
$z = 1$

over domain D, we'll get (in cubic units)

The same as the area (in square units) of D.

Ex. Find the area of one leaf of the lemniscate

$r^2 = 4\cos(2\theta)$



note: only exists

where $4\cos(2\theta) \geq 0$

ie $-\frac{\pi}{2} \leq 2\theta \leq \frac{\pi}{2}$, $\frac{3\pi}{2} \leq 2\theta \leq \frac{5\pi}{2}$, ...

$$\underbrace{-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}}, \quad \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$$

defines the whole graph since $r = \pm\sqrt{4\cos 2\theta}$

one leaf where $r = \sqrt{4\cos 2\theta}$ over $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

$$\int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta = 2\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta = 2\int_0^{\pi/4} \frac{r^2}{2}\Big|_0^{\sqrt{4\cos 2\theta}} d\theta$$

$$= 2\int_0^{\pi/4} \frac{4\cos 2\theta}{2}\,d\theta = 4\cdot\frac{1}{2}\sin 2\theta\Big|_0^{\pi/4}$$

$$= 2\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= 2.$$

In Calc II, area $= \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2\,d\theta$

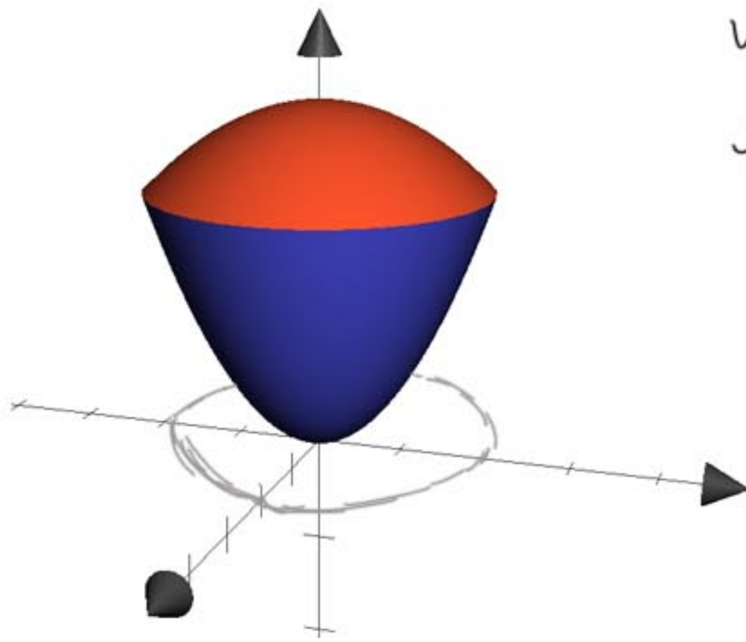
for $r = f(\theta)$
$= \sqrt{4\cos 2\theta}$

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} 4\cos 2\theta\,d\theta = \int_0^{\pi/4} 4\cos 2\theta\,d\theta$$

= Same.

Ex. Find the volume of the solid between

$$z = 16 - x^2 - y^2 \quad \text{and} \quad z = 3x^2 + 3y^2$$



we see from the graph that the curve of intersection will help us describe the domain

$$16 - x^2 - y^2 = 3x^2 + 3y^2$$

$$16 = 4x^2 + 4y^2$$

$$4 = x^2 + y^2$$

so $D = \{(x,y) : x^2 + y^2 \le 4\}$

$$\text{volume} = \iint_D \left((16 - x^2 - y^2) - (3x^2 + 3y^2) \right) dA$$

$$= \iint_D (16 - 4x^2 - 4y^2)\, dA = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} (16 - 4x^2 - 4y^2)\, dy\, dx$$

OR

$$\int_0^{2\pi}\int_0^2 (16-4r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (16r-4r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[8r^2 - r^4\right]_0^2 d\theta = \int_0^{2\pi}(32-16)\,d\theta$$

$$= \int_0^{2\pi} 16\,d\theta = 16\theta\Big|_0^{2\pi} = 32\pi.$$