# Math 20300
# Calculus III

# Lesson 18
# Relative Extrema and Saddle Points

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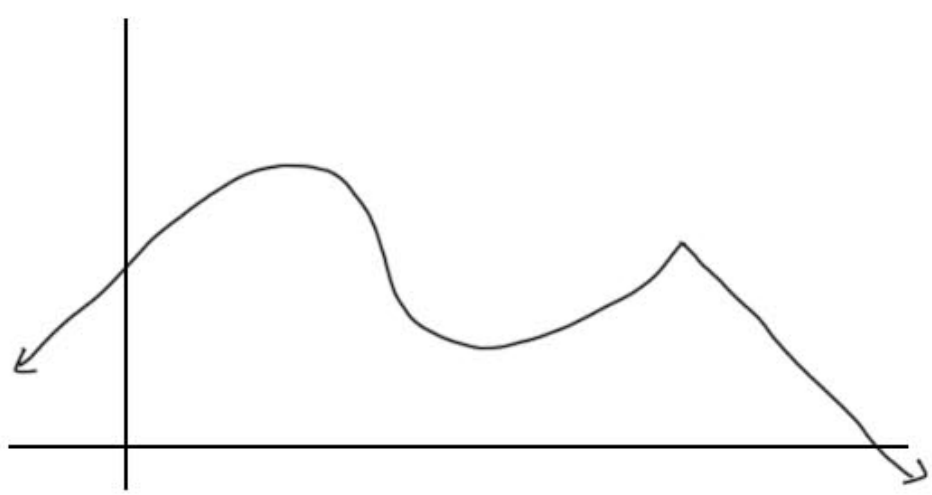
# Relative Extrema and Saddle Points

Recall from Calc I:

A function $f$ has a <u>local/relative maximum</u> at $x=c$ if $f(x) \le f(c)$ $\forall x$ in some open interval containing $c$.

A function $f$ has a <u>local/relative minimum</u> at $x=c$ if $f(x) \ge f(c)$ $\forall x$ in some open interval containing $c$.

At points of relative max or min, either $f'(c)=0$ or $f'(c)$ does not exist



We say $x=c$ is a <u>critical point</u> of $f$ if

$x=c$ is in the domain of $f$ and

if $f'(c)=0$ or $f'(c)$ does not exist.

If $f'(c)=0$, the tangent line to $f(x)$ at $x=c$ has slope $=0$, ie the tangent line is horizontal.

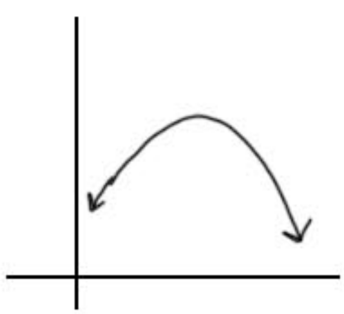
To find relative extrema, we find the critical points of $f$ and test them.

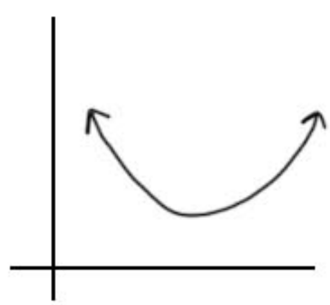
Second Derivative Test for Relative Extrema:

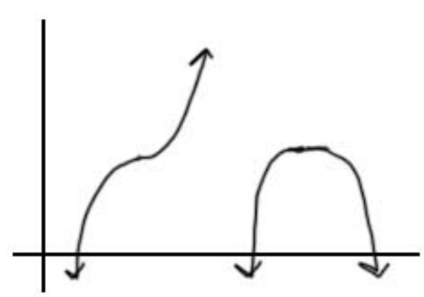
if $f'(c)=0$ and $f''(c)<0$, local max at $x=c$

if $f'(c)=0$ and $f''(c)>0$, local min at $x=c$

if $f'(c)=0$ and $f''(c)=0$, no conclusion from this test.





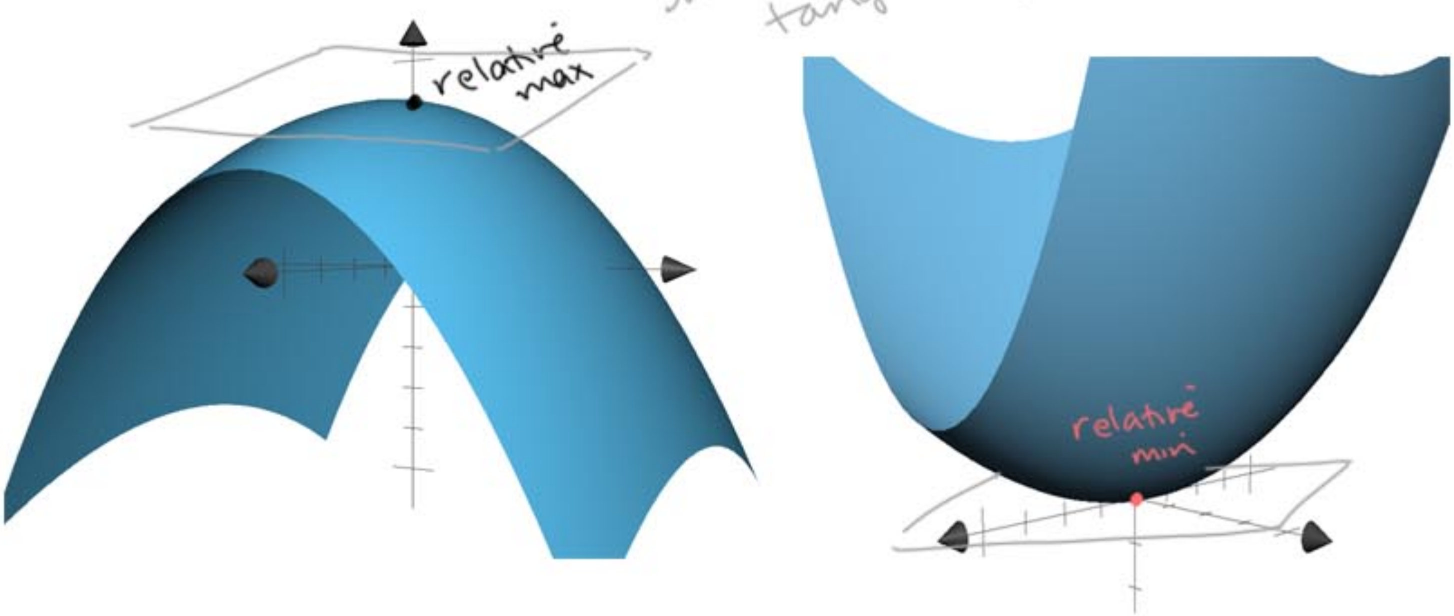


Now for $z = f(x,y)$,

$f$ has a <u>local/relative maximum</u> at $(a,b)$ if $f(x,y) \le f(a,b)$ $\forall (x,y)$ in some open disk around $(a,b)$. And $f(a,b)$ is <u>the local/relative maximum value.</u>

$f$ has a <u>local/relative minimum</u> at $(a,b)$ if $f(x,y) \ge f(a,b)$ $\forall (x,y)$ in some open disk around $(a,b)$. And $f(a,b)$ is <u>the local/relative minimum value.</u>

horizontal tangent plane since $f_x(0,0) = f_y(0,0) = 0$



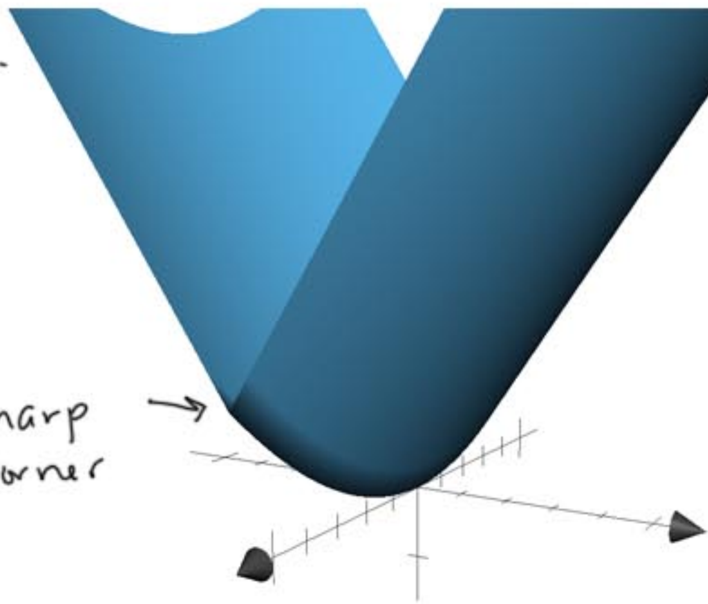
The point $(a,b)$ is a <u>critical point</u> of $f$ if

$(a,b)$ is in the domain of $f$ and if

either $\quad f_x(a,b) = 0$ and $f_y(a,b) = 0$

or one or both of $f_x(a,b)$, $f_y(a,b)$ do not exist.

Ex.



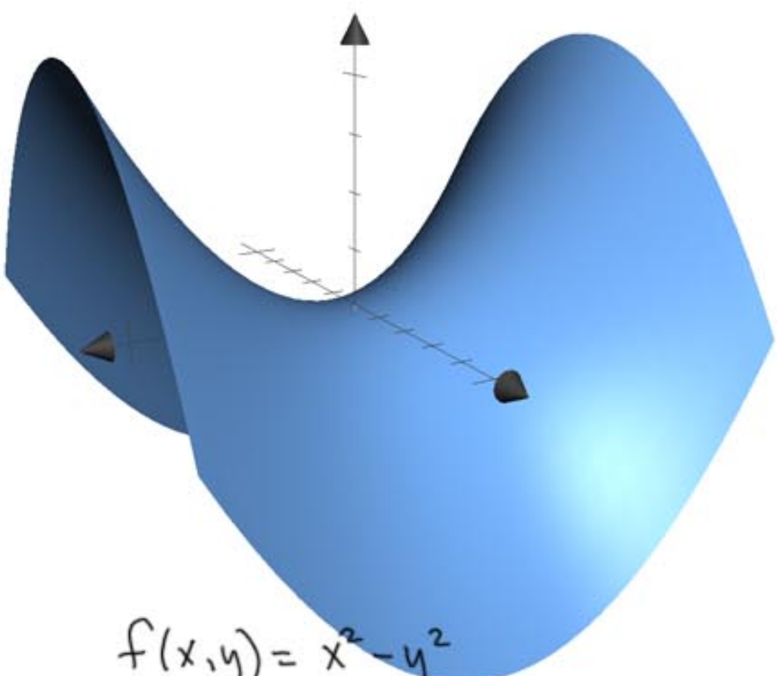
sharp corner

$f(x,y) = x^2 + 10|y|$

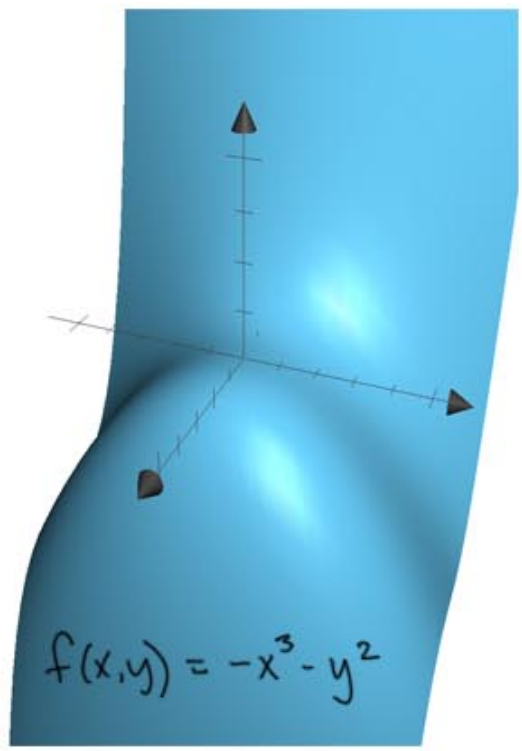
$f_y$ does not exist when $y = 0$, that gives a whole line of critical points

If both $f_x(a,b) = 0$ and $f_y(a,b) = 0$, the tangent plane at $(a, b, f(a,b))$ is horizontal

A differentiable function $f$ has a <u>saddle point</u> at $(a,b)$ if $(a,b)$ is a critical point of $f$ and if in every open disk centered at $(a,b)$ there are points $(x,y)$ in the domain of $f$ with $f(x,y) > f(a,b)$ and points $(x,y)$ in the domain of $f$ with $f(x,y) < f(a,b)$. The point $(a, b, f(a,b))$ is called a saddle point of the surface.



$f(x,y) = x^2 - y^2$



$f(x,y) = -x^3 - y^2$

To find relative extrema and saddle points of a function of two variables, we find critical points and test them:

## The Second Derivative Test for Relative Extrema (and saddle points):

Suppose $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centered at $(a,b)$ and that $f_x(a,b) = f_y(a,b) = 0$.

Let $D = D(a,b) = f_{xx}(a,b) \cdot f_{yy}(a,b) - (f_{xy}(a,b))^2$

if $D > 0$ and $f_{xx}(a,b) < 0$, $f(a,b)$ is a local maximum.

if $D > 0$ and $f_{xx}(a,b) > 0$, $f(a,b)$ is a local minimum.

if $D < 0$, $f$ has a saddle point at $(a,b)$.

if $D = 0$, the test is inconclusive.

$D$ is called the discriminant of $f$.

Ex. $f(x,y) = x^2 + 4y^2$ find all local extrema and saddle points.

So we first find critical points, then test them.

$f_x(x,y) = 2x$ $\qquad$ $f_y(x,y) = 8y$

$8y = 0$

$2x = 0$

$x = 0$ $\quad$ AND $\quad$ $y = 0$

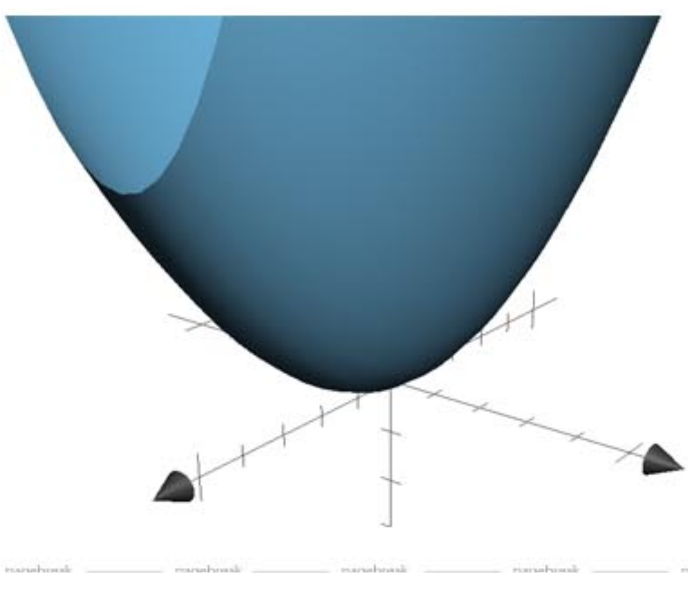
only critical point $(0,0)$

$f_{xx}(x,y) = 2$ $\quad$ $f_{yy}(x,y) = 8$ $\quad$ $f_{xy}(x,y) = 0$

$f_{xx}(0,0) = 2$ $\quad$ $f_{yy}(0,0) = 8$ $\quad$ $f_{xy}(0,0) = 0$

$$\left.\begin{array}{l} D = 2 \cdot 8 - 0^2 = 16 > 0 \\ \text{and } f_{xx}(0,0) = 2 > 0 \end{array}\right\} \text{local minimum at } (0,0)$$

no other local extrema or saddles.



$f(x,y) = x^2 + 4y^2$

(elliptic paraboloid)

Ex. $f(x,y) = x^2 - y^2$ find all local extrema and saddle points


Work on this problem on your own

$f_x(x,y) = 2x \qquad f_y(x,y) = -2y$

$2x = 0 \qquad -2y = 0$

$x = 0$ AND $y = 0$

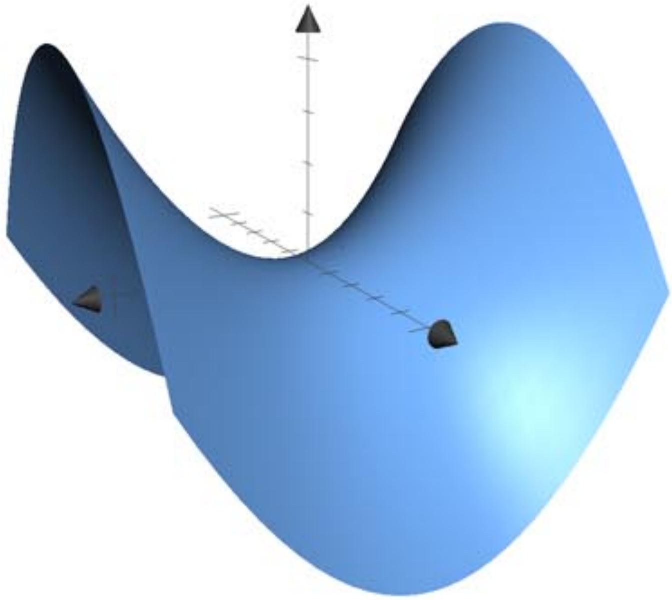
$(0,0)$ is the only critical point.

$f_{xx}(x,y) = 2 \qquad f_{yy}(x,y) = -2 \qquad f_{xy}(x,y) = 0$

$f_{xx}(0,0) = 2 \qquad f_{yy}(0,0) = -2 \qquad f_{xy}(0,0) = 0$

$$D = 2(-2) - 0^2 = -4 < 0$$

$\therefore$ $f$ has a saddle point at $(0,0)$



$f(x,y) = x^2 - y^2$

(hyperbolic paraboloid)

Ex. $f(x,y) = x^4 + y^4 + 4xy$ find all local extrema and saddle points

$f_x(x,y) = 4x^3 + 4y \qquad f_y(x,y) = 4y^3 + 4x$

$4x^3 + 4y = 0 \quad$ AND $\quad 4y^3 + 4x = 0$

solve for y

$y = -x^3$ sub into ↗ $\quad 4(-x^3)^3 + 4x = 0$

$(-x^3)^3 + x = 0$

$-x^9 + x = 0$

$x(-x^8 + 1) = 0$

$x = 0 \qquad x^8 = 1 \qquad x = \pm\sqrt[8]{1}$

$x = \pm 1$

and for each of These x-values, The $y = -x^3$

$x = 0, \quad y = -(0)^3 = 0 \qquad x = 1, \; y = -(1)^3 = -1$

$(0,0) \qquad\qquad (1,-1)$

$x = -1 \quad y = -(-1)^3 = +1$

$(-1,1)$

$f_x(x,y) = 4x^3 + 4y \qquad f_y(x,y) = 4y^3 + 4x$

$f_{xx}(x,y) = 12x^2 \qquad f_{yy}(x,y) = 12y^2 \qquad f_{xy} = 4$

$f_{xx}(0,0) = 0 \qquad f_{yy}(0,0) = 0 \qquad f_{xy}(0,0) = 4$

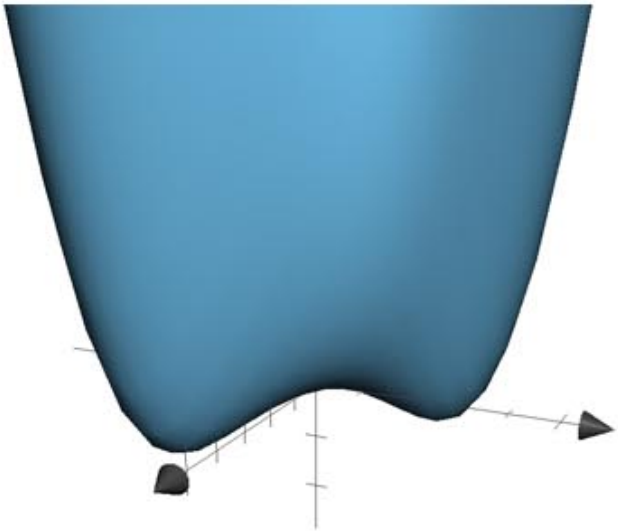
$D(0,0) = 0 \cdot 0 - 4^2 = -16 < 0 \Rightarrow$ saddle pt at $(0,0)$.

$f_{xx}(1,-1) = 12 \qquad f_{yy}(1,-1) = 12 \qquad f_{xy} = 4$

$$\left.\begin{array}{l} D(1,-1) = 12 \cdot 12 - 4^2 = 144 - 16 > 0 \\ f_{xx}(1,-1) > 0 \end{array}\right\} \text{local minimum at } (1,-1)$$

$f_{xx}(-1,1) = 12 \qquad f_{yy}(-1,1) = 12 \qquad f_{xy} = 4$

$$\left.\begin{array}{l} D(-1,1) = 12 \cdot 12 - 4^2 = 144 - 16 > 0 \\ f_{xx}(-1,1) > 0 \end{array}\right\} \text{local minimum at } (-1,1)$$



$f(x,y) = x^4 + y^4 + 4xy$

saddle at $(0,0,0)$

min at $(1,-1,-2)$

min at $(-1,1,-2)$

Ex. Compare above to $f(x,y) = x^2 + y^4 - 2y^2$

$f_x(x,y) = 2x \qquad f_y(x,y) = 4y^3 - 4y$

$2x = 0 \qquad 4y^3 - 4y = 0$

$x = 0$ AND $\quad y^3 - y = 0$

$y(y^2 - 1) = 0$

$y = 0, \; y^2 - 1 = 0$

$y = \pm 1$

$(0,0) \quad (0,1) \quad (0,-1)$ are the critical pts.

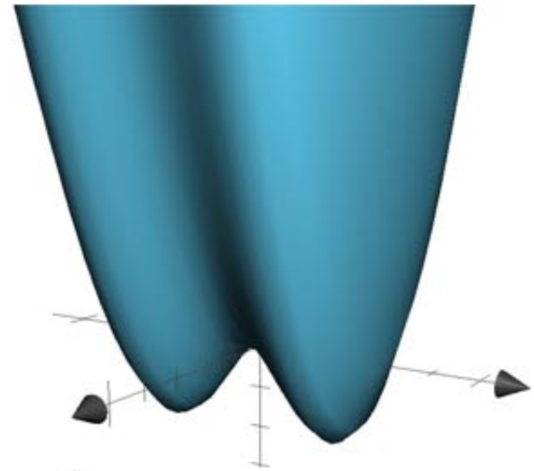
$f_{xx}(x,y) = 2 \qquad f_{yy}(x,y) = 12y^2 - 4 \qquad f_{xy}(x,y) = 0$

$D(0,0) = 2(-4) - 0^2 = -8 < 0$ saddle pt at $(0,0)$

$D(0,1) = 2(8) - 0^2 = 16 > 0$ and $f_{xx}(0,1) > 0 \Rightarrow$ local min at $(0,1)$

$D(0,-1) = 2(8) - 0^2 = 16 > 0$

and $f_{xx}(0,-1) > 0 \Rightarrow$
local min at $(0,-1)$



$f(x,y) = x^2 + y^4 - 2y^2$

Ex. $f(x,y) = \sin(x^2+y^2)$ find all local extrema and saddle points

$f_x(x,y) = \cos(x^2+y^2)\cdot 2x$ $\quad$ $f_y(x,y) = \cos(x^2+y^2)\cdot 2y$

$= 2x\cos(x^2+y^2)$ $\quad$ $= 2y\cos(x^2+y^2)$

need $2x\cos(x^2+y^2) = 0$ AND $2y\cos(x^2+y^2) = 0$

either $\cos(x^2+y^2) = 0$ or, if $\cos(x^2+y^2) \neq 0$,

need $\underbrace{2x = 0 \text{ and } 2y = 0}_{x=0,\ y=0}$

$(0,0)$

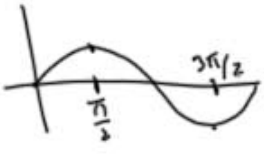
if $\cos(x^2+y^2) = 0$, $x^2+y^2 = (2k+1)\frac{\pi}{2}$ $\quad k \in \mathbb{Z}$

odd multiples of $\frac{\pi}{2}$.

can also write as $x^2+y^2 = \frac{\pi}{2} + 2\pi k$, or

$x^2+y^2 = \frac{3\pi}{2} + 2\pi k$ for $k \in \mathbb{Z}$

this is preferable, since $\sin(x^2+y^2)$ will be different for $x^2+y^2 = \frac{\pi}{2} + 2\pi k$ and $x^2+y^2 = \frac{3\pi}{2} + 2\pi k$




notice These are circles of critical points in the domain $\mathbb{R}^2$

$f_x(x,y) = 2x\cos(x^2+y^2)$ $\quad f_y(x,y) = 2y\cos(x^2+y^2)$

$f_{xx}(x,y) = 2\cdot\cos(x^2+y^2) + 2x(-\sin(x^2+y^2))(2x)$

$= 2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2)$

$f_{yy}(x,y) = 2\cos(x^2+y^2) - 4y^2\sin(x^2+y^2)$

$f_{xy}(x,y) = 2x\left(-\sin(x^2+y^2)\right)2y$

$= -4xy\sin(x^2+y^2)$

so $D(0,0) = (2)(2) - 0^2 = 4 > 0$ $\quad f_{xx}(0,0) > 0 \Rightarrow$ local min at $(0,0)$

Now for circles $x^2+y^2 = \frac{\pi}{2} + 2\pi k \quad k \in \mathbb{Z}$,

we know $\sin(x^2+y^2) = 1$, $\cos(x^2+y^2) = 0$

so $f_{xx}(x,y) = 2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2)$

$0 - 4x^2(1) = -4x^2$

$f_{yy}(x,y) = 2\cos(x^2+y^2) - 4y^2\sin(x^2+y^2)$

$0 - 4y^2(1) = -4y^2$

$f_{xy}(x,y) = -4xy\sin(x^2+y^2)$

$= -4xy(1) = -4xy$

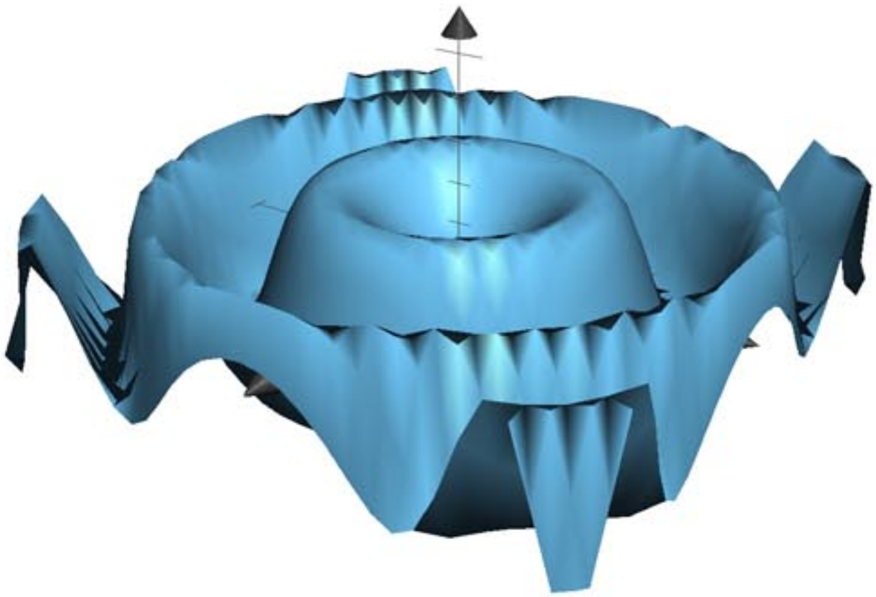
$\therefore D = (-4x^2)(-4y^2) - (-4xy)^2$

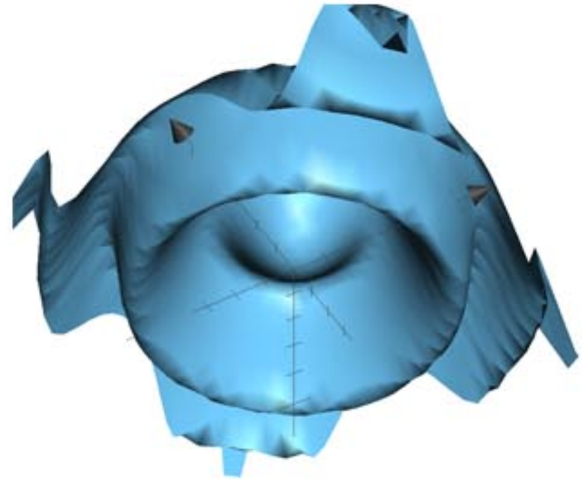
$16x^2y^2 - 16x^2y^2 = 0$. Second Deriv test is inconclusive.

Similar computation for $x^2+y^2 = \frac{3\pi}{2} + 2\pi k$

except $\sin(x^2+y^2) = -1$

look at graph for more info:





view from underneath

local maximums at points $(x,y)$ with $x^2+y^2 = \frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$

local min at $(0,0,0)$

local minimums at points $(x,y)$ with $x^2+y^2 = \frac{3\pi}{2} + 2\pi k$ $k \in \mathbb{Z}$

Additional comments: $f(x,y) = -x^3 - y^2$ (above) has a critical pt at $(0,0)$, but $D = 0$. Looking at graph we see $(0,0,0)$ is a saddle point of the surface.