Math 20300
Calculus III

Lesson 17
Directional Derivatives and The Gradient Vector

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Directional Derivatives

Recall from Lesson 14:

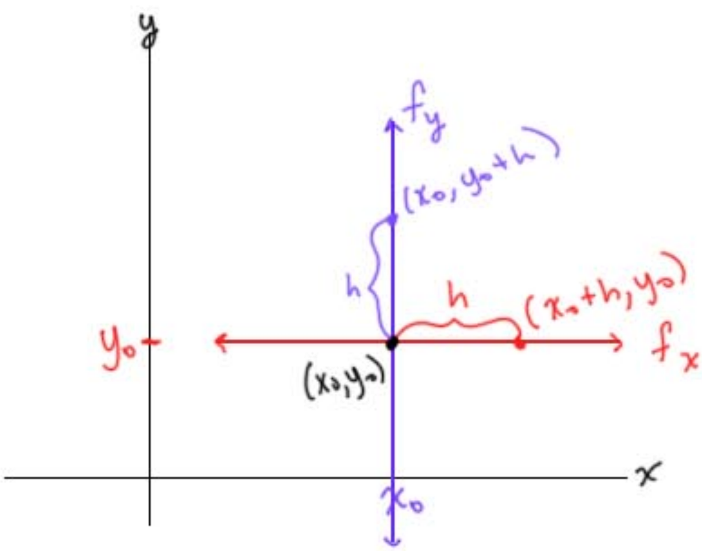
Definitions:

y_0 is fixed

$$f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,\, y_0) - f(x_0, y_0)}{h}$$

x_0 is fixed

$$f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,\, y_0+h) - f(x_0, y_0)}{h}$$

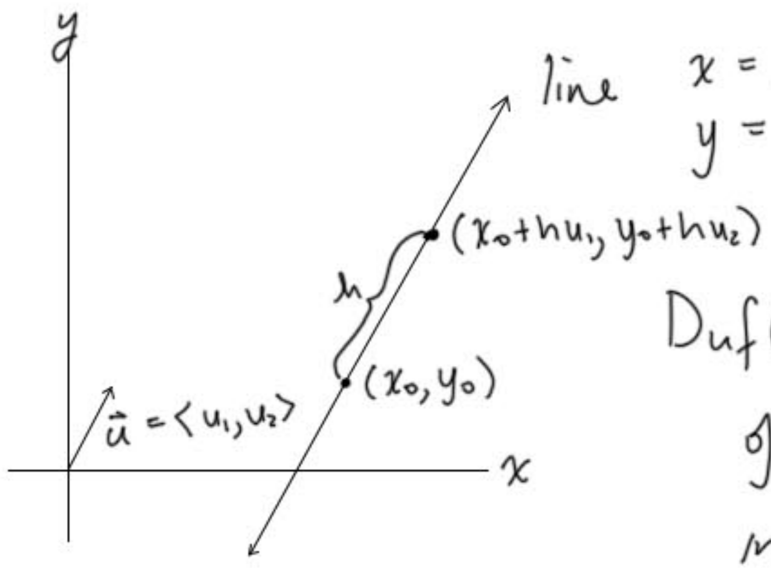


The $\underline{\text{directional derivative of } f(x,y)}$ at (x_0, y_0)

in the direction of unit vector $\vec{u} = \langle u_1, u_2 \rangle$

is $$D_{\vec{u}} f(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0+hu_1,\ y_0+hu_2) - f(x_0,y_0)}{h}$$

(provided the limit exists).



$D_u f(x_0,y_0)$ gives the rate of change of f at (x_0,y_0) in direction $\langle u_1, u_2 \rangle$

Notice, if $\vec{u} = \vec{i} = \langle 1, 0 \rangle$

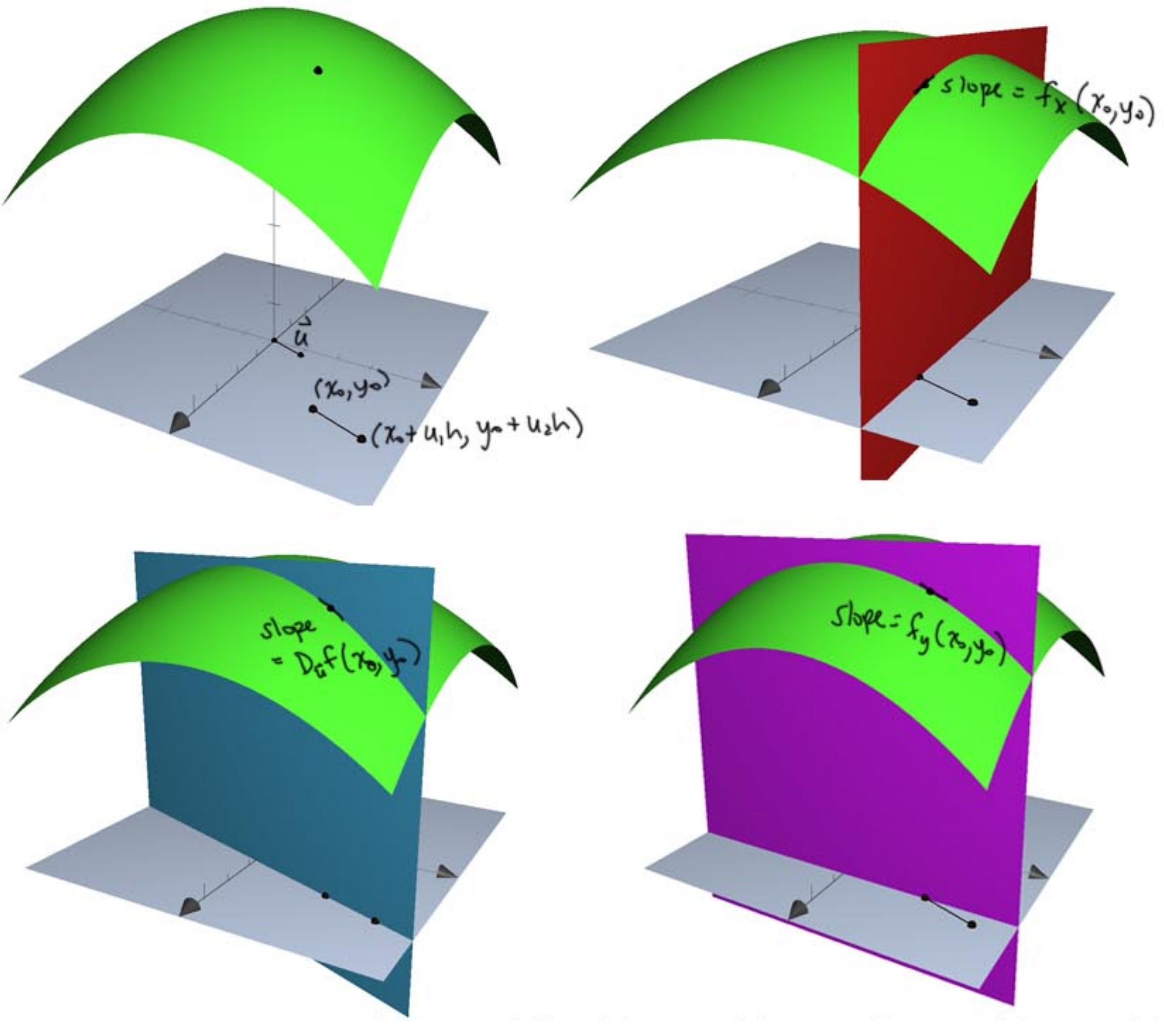
$$D_{\vec{u}} f(x_0,y_0) = \lim_{h \to 0} \frac{f(x_0+hu_1,\ y_0+hu_2) - f(x_0,y_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(x_0+h,\ y_0) - f(x_0,y_0)}{h} = f_x(x_0,y_0)$$

and if $\vec{u} = \vec{j} = \langle 0,1 \rangle$

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} = f_y(x_0, y_0)$$



Above we have the limit definition of $D_{\vec{u}}f$, but how do we compute it?

Theorem: If f is a differentiable function of x & y, then f has a directional derivative in direction of any unit vector $\vec{u} = \langle u_1, u_2 \rangle$ and

$$D_{\vec{u}}f(x,y) = f_x(x,y)\,u_1 + f_y(x,y)\,u_2$$

Ex. $f(x,y) = 4 - x^2 - y^2 \qquad \vec{u} = \langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle$

find $D_{\vec{u}}f(\frac{1}{2}, \frac{1}{2})$

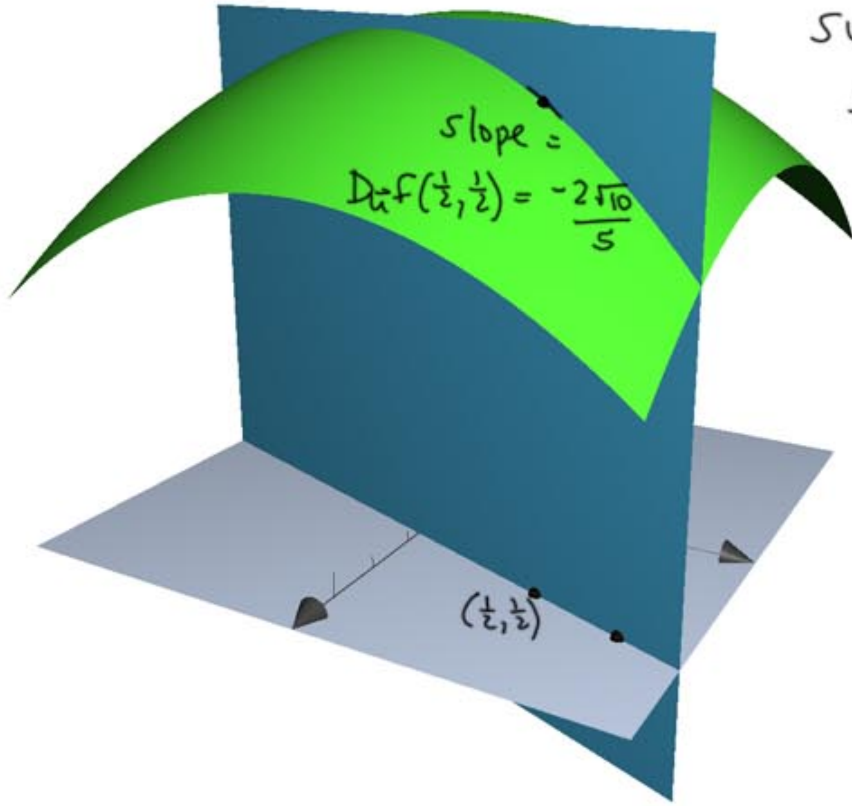
$$D_{\vec{u}}f(x,y) = -2x\left(\frac{1}{\sqrt{10}}\right) + -2y\left(\frac{3}{\sqrt{10}}\right)$$

$$= \frac{-2}{\sqrt{10}}x - \frac{6}{\sqrt{10}}y$$

$$D_{\vec{u}}f(\tfrac{1}{2}, \tfrac{1}{2}) = \frac{-2}{\sqrt{10}}\left(\frac{1}{2}\right) - \frac{6}{\sqrt{10}}\left(\frac{1}{2}\right) = -\frac{1}{\sqrt{10}} - \frac{3}{\sqrt{10}}$$

$$= -\frac{4}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}}$$

$$= \frac{-2\sqrt{10}}{5}$$



Def. If f is a function of x and y, the <u>gradient of f</u> is the vector function

$$\vec{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

$$= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

Notice then, $D_{\vec{u}} f(x,y) = \vec{\nabla} f(x,y) \cdot \vec{u}$

Ex. Find the directional derivative of

$$f(x,y) = 3x + 2xy^2 - 6\cosh(xy) \quad \text{at } (2,1)$$

heading toward the point $(1,2)$.

we need to first find unit vector $\vec{u} = \langle u_1, u_2 \rangle$

giving the direction from $(2,1)$ to $(1,2)$

not necessarily a unit vector $\rightarrow$ $\vec{v} = \langle 1-2,\ 2-1 \rangle = \langle -1, 1 \rangle$

$$\|\vec{v}\| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

$$\vec{u} = \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \quad \text{unit vector.}$$

$$D_{\vec{u}} f(2,1) = f_x(2,1)\left(\frac{-1}{\sqrt{2}}\right) + f_y(2,1)\left(\frac{1}{\sqrt{2}}\right)$$

we have $f(x,y) = 3x + 2xy^2 - 6\cosh(xy)$

$$f_x(x,y) = 3 + 2y^2 - 6\sinh(xy) \cdot y$$

$$= 3 + 2y^2 - 6y\sinh(xy)$$

$\frac{d}{dx}(\cosh x) = \sinh x$

$$f_x(2,1) = 3 + 2 - 6\sinh(2) = 5 - 6\sinh(2)$$

$$f_y(x,y) = 4xy - 6x\sinh(xy)$$

$$f_y(2,1) = 4(2)(1) - 6(2)\sinh(2) = 8 - 12\sinh(2)$$

so $D_{\vec{u}} f(2,1) = f_x(2,1)\left(-\frac{1}{\sqrt{2}}\right) + f_y(2,1)\left(\frac{1}{\sqrt{2}}\right)$

$$= \left(5 - 6\sinh(2)\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(8 - 12\sinh(2)\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(8 - 5 - 12\sinh 2 + 6\sinh 2\right)$$

$$= \frac{1}{\sqrt{2}}\left(3 - 6\sinh(2)\right).$$

More about the gradient $\vec{\nabla} f(x,y)$ and the directional derivative:

Which direction would maximize the directional derivative? (give the most change in z, ie the steepest slope)

$$D_{\vec{u}}f = \vec{\nabla}f \cdot \vec{u} = \|\vec{\nabla}f\| \underbrace{\|\vec{u}\|}_{1} \cos\theta$$

where θ is the angle between $\vec{\nabla}f$ and $\vec{u}$

$$\therefore \quad D_{\vec{u}}f = \|\vec{\nabla}f\| \underbrace{\cos\theta}$$

maximized when $\theta = 0$, $\cos\theta = 1$

$\theta = 0$ means $\vec{u}$ is in the same direction as $\vec{\nabla}f$.

and then the maximum value of $D_{\vec{u}}f$,

ie the steepest slope is $\|\vec{\nabla}f\|$.

Also note $D_{\vec{u}}f = \|\vec{\nabla}f\| \cos\theta = 0$ when $\theta = \frac{\pi}{2}$

so when $\vec{u}$ is orthogonal to $\vec{\nabla}f$ (angle is $\frac{\pi}{2}$)

there is $\underbrace{\text{no change in } f}$

$\vec{u}$ is tangent to level curves of f

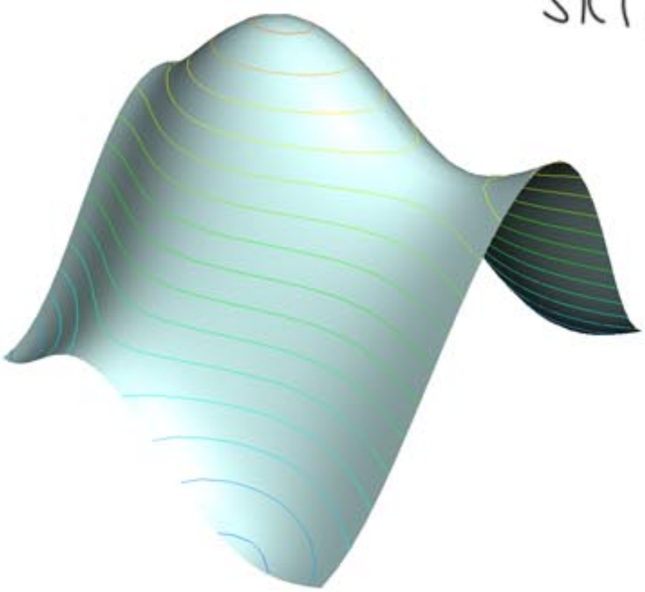
$\therefore$ $\vec{\nabla}f$ is orthogonal to level curves of f
$(f(x,y) = k)$

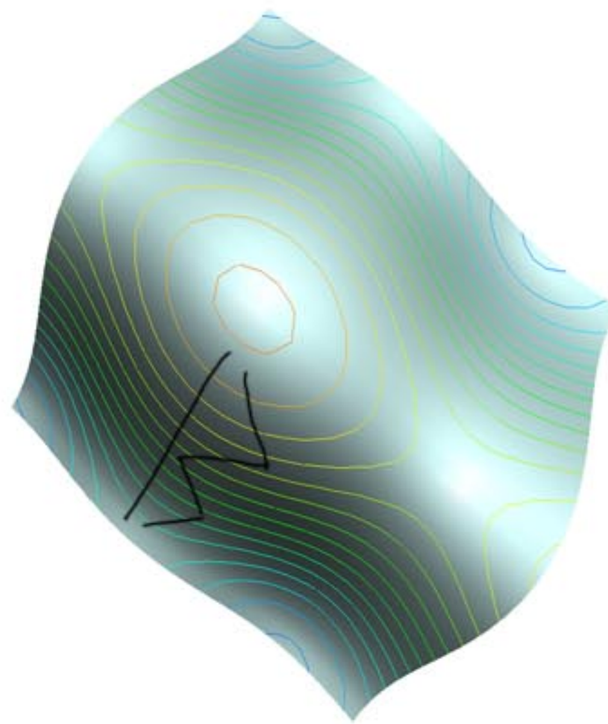
$\therefore$ 1) gradient $\vec{\nabla} f$ points in the direction of maximum increase of f

2) length of gradient $\|\vec{\nabla} f\|$ is that maximum slope.

3) gradient $\vec{\nabla} f$ is orthogonal to level curves of f.

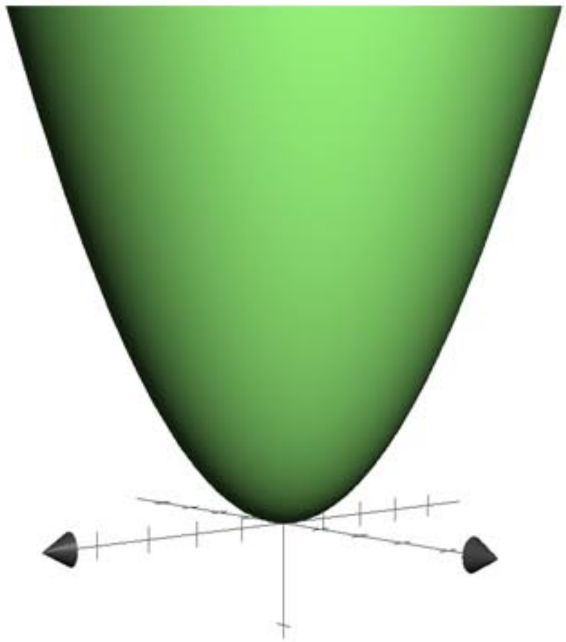
Skiing

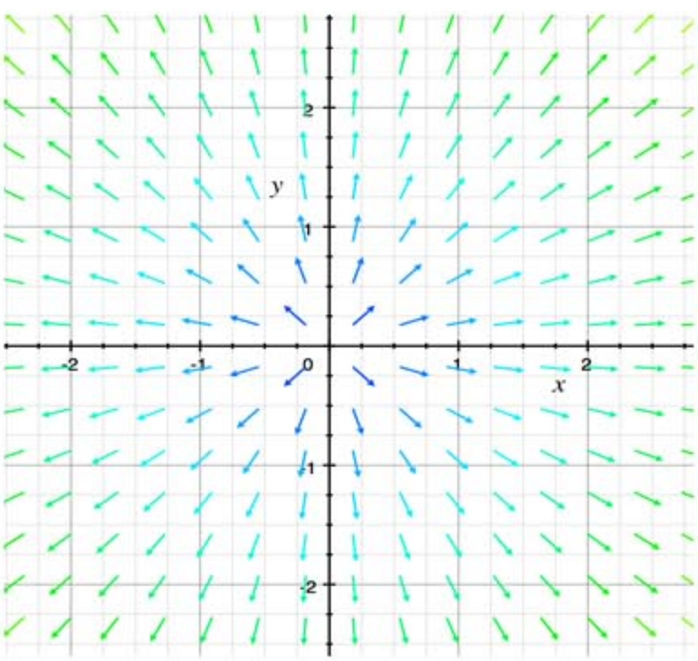




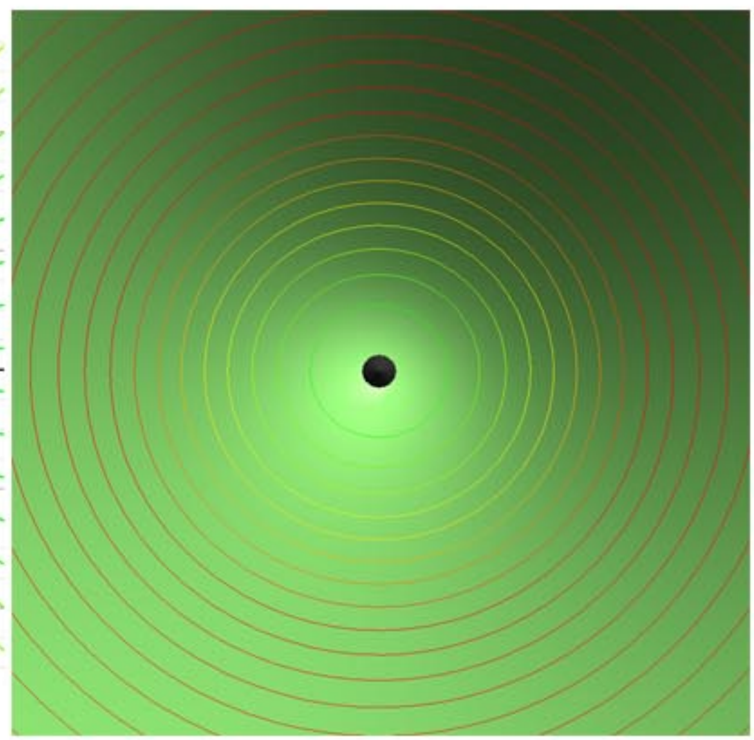
less steep not to go straight down the mountain

Consider $z = x^2 + y^2$ elliptic paraboloid



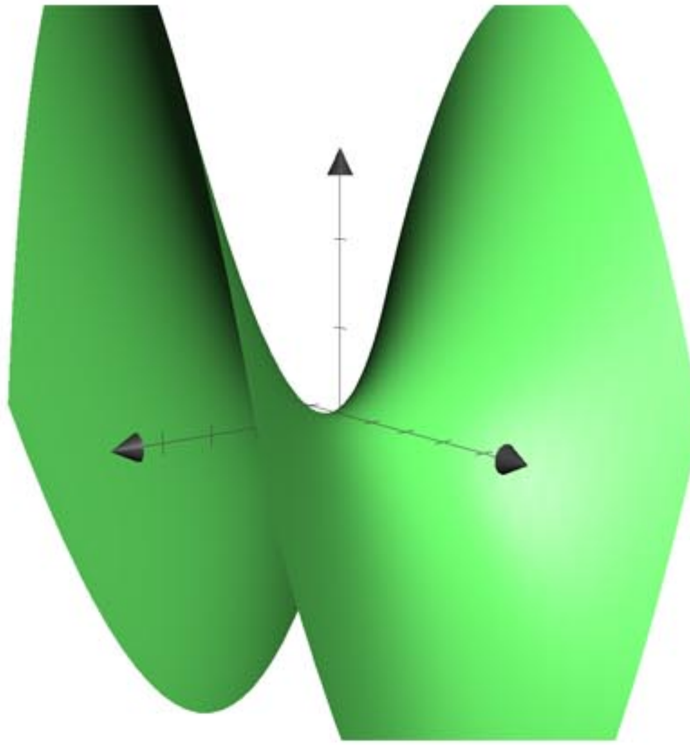


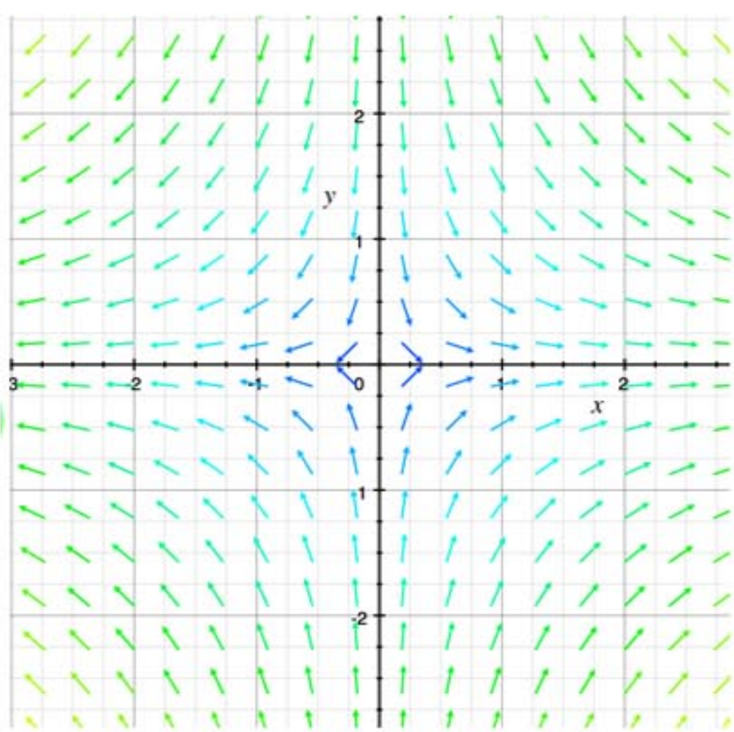
gradient vector field



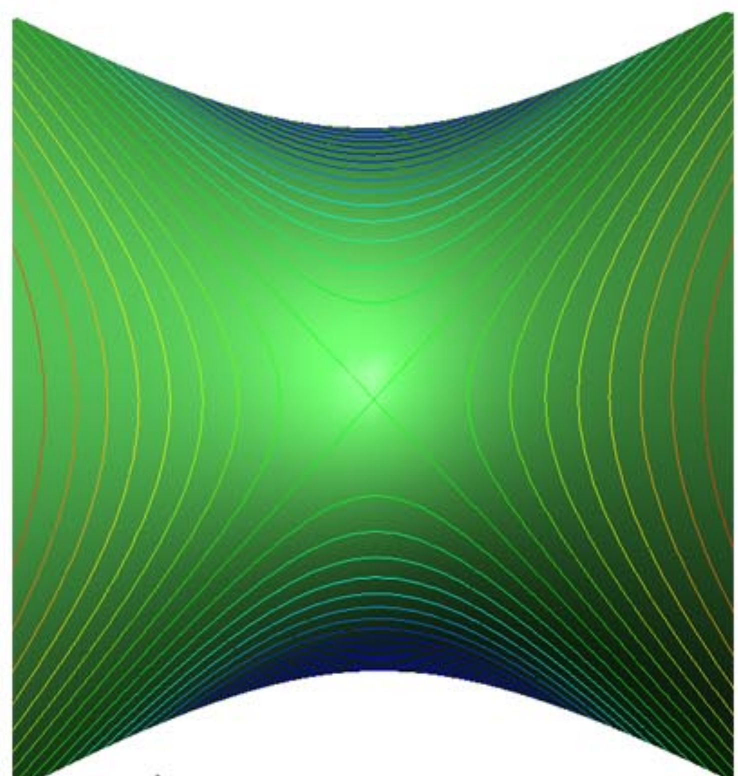
level curves

$z = x^2 - y^2$ hyperbolic paraboloid





gradient vector field



level curves

gradient vector $\perp$ level curves $f(x,y)=k$

$\therefore$ gradient vector $\perp$ tangent lines to level curves

Recall level surfaces from lesson 11:

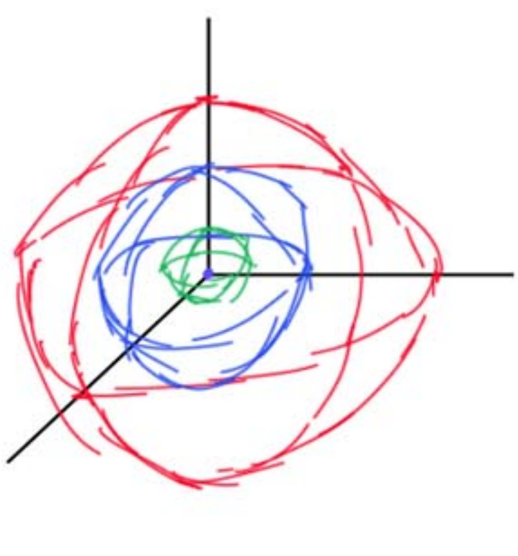
$$\underbrace{f(x,y,z)}_{\text{set = constant}} = x^2+y^2+z^2$$

$1 = x^2+y^2+z^2$

$4 = x^2+y^2+z^2$ spheres of different radii

$9 = x^2+y^2+z^2$

$0 = x^2+y^2+z^2$ just the point $(0,0,0)$



level surfaces

for $w=f(x,y,z)$, we have $\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$ and

gradient vector is $\perp$ level surfaces $f(x,y,z)=k$

$\therefore$ gradient vector is $\perp$ tangent planes to level surfaces proof:

Consider level surface $f(x,y,z) = k$, and any unit vector $\vec{u}$ in the tangent plane to the surface at (x_0, y_0, z_0). Then since $\vec{u}$ is in the tangent plane, $D_{\vec{u}} f(x_0, y_0, z_0) = 0$ (since f is constant on level surfaces).

$$0 = D_{\vec{u}} f(x_0, y_0, z_0) = \vec{\nabla} f(x_0, y_0, z_0) \cdot \vec{u}$$

$\therefore$ $\vec{\nabla} f(x_0, y_0, z_0)$ and $\vec{u}$ are orthogonal.

$\therefore$ $\vec{\nabla} f(x_0, y_0, z_0)$ is normal to the tangent plane + normal to the surface.

Ex above $f(x,y,z) = x^2 + y^2 + z^2$

For level surface $x^2 + y^2 + z^2 = 9$

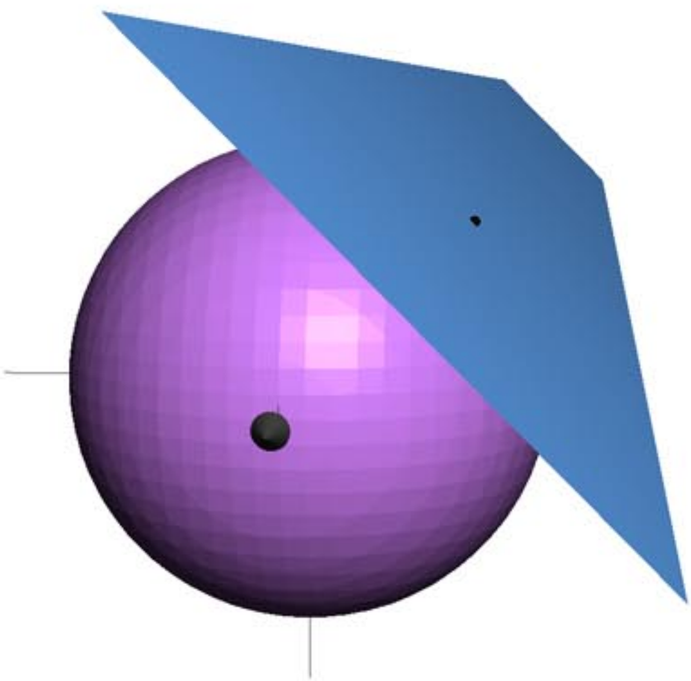
consider the tangent plane at $(1, 2, 2)$.

we now have that $\vec{\nabla} f(1,2,2)$ is normal to the tangent plane

$\vec{\nabla} f(x,y,z) = \langle 2x, 2y, 2z \rangle$

$\vec{\nabla} f(1,2,2) = \langle 2,4,4 \rangle$ normal to tangent plane at $(1,2,2)$

$2(x-1) + 4(y-2) + 4(z-2) = 0$



Ex. Find an equation of the tangent plane to the surface given by $x^2 + 4y^2 + 17 = z^2$ at $(-2,1,5)$.

Consider $F(x,y,z) = x^2+4y^2-z^2+17$

then we want a tangent plane to the level surface $F(x,y,z)=0$ at $(-2,1,5)$.

$\vec{\nabla}F(x,y,z) = \langle 2x, 8y, -2z\rangle$

$\vec{\nabla}F(-2,1,5) = \langle -4, 8, -10\rangle$

$-4(x+2)+8(y-1)-10(z-5)=0$.

Can also solve by methods of lesson 15:

In lesson 15 we found that

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0$$

is an equation for the tangent plane to $z=f(x,y)$ at (x_0,y_0). Notice we'd need implicit differentiation to get $f_x = \frac{\partial z}{\partial x}$ and $f_y = \frac{\partial z}{\partial y}$.

we get $\frac{\partial z}{\partial x} = \frac{x}{z}$ $\quad \frac{\partial z}{\partial y} = \frac{4y}{z}$

So $\left.\frac{\partial z}{\partial x}\right|_{(-2,1,5)} = \frac{-2}{5}$ and $\left.\frac{\partial z}{\partial y}\right|_{(-2,1,5)} = \frac{4}{5}$.

Then the tangent plane can be written as

$$\frac{-2}{5}(x+2) + \frac{4}{5}(y-1) - (z-5) = 0$$

(multiply by 10, same equation as above.)