Math 20300
Calculus III

Lesson 13
Continuity

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Continuity

for $y = f(x)$, we said $f(x)$ is continuous at $x = c$ if :

1) $\lim\limits_{x \to c} f(x)$ exists
2) $f(c)$ exists
3) $\lim\limits_{x \to c} f(x) = f(c)$

(notice, it is enough to say 3) as that assumes both sides of = sign exist).

for $z = f(x,y)$, <u>f is continuous at (a,b)</u>

if $\lim\limits_{(x,y) \to (a,b)} f(x,y) = f(a,b)$. (this assumes both sides exist.)

We say <u>f is continuous on D</u> if for every $(a,b) \in D$, f is continuous at (a,b).

For $f(x,y)$ + $g(x,y)$ continuous at (a,b)

$$\left.\begin{array}{l} f+g \\ f-g \\ f\cdot g \\ \dfrac{f}{g} \end{array}\right\} \text{are continuous at } (a,b)$$

$g(a,b) \neq 0$

For $f(x,y)$ continuous at (a,b) and

$g(x)$ continuous at $f(a,b)$,

we have $g \circ f$ continuous at (a,b)

$$g(f(x,y))$$

Functions that are continuous:

polynomial functions

rational functions (where they exist)

A <u>polynomial function of two variables</u> is a sum of terms of the form $cx^m y^n$

c constant, m, n non-negative integers.

Ex. $f(x,y) = 3x^2y + 2xy - x$

$g(x,y) = 5x^{17}y^{23} - 4x^{13}y^9 + 3x + 5$

Since polynomials are continuous, we can take limits by just plugging in

$$\lim_{(x,y)\to(2,-3)} \underbrace{3x^2y + 2xy - x}_{f(x,y)} = f(2,-3)$$

$$= 3(2)^2(-3) + 2(2)(-3) - 2$$

$$= -36 - 12 - 2 = -50.$$

A <u>rational function of two variables</u>

is of the form

$$f(x,y) = \frac{p(x,y)}{q(x,y)}$$ both polynomials

defined where $q(x,y) \neq 0$.

Since rational functions are continuous

where they are defined, we can take

limits by plugging in (as long as $q(a,b)$ exists).

$$\lim_{(x,y)\to(-1,4)} \frac{2x^2y - xy^2 + x}{x^4y - 2xy + 1} = \frac{2(-1)^2(4) - (-1)4^2 + -1}{(-1)^4(4) - 2(-1)4 + 1}$$

$$= \frac{8 - 16 - 1}{4 + 8 + 1} = \frac{-9}{13}.$$

Since we said $g(f(x,y))$ is continuous for $f(x,y)$ + $g(x)$ continuous, we get lots of continuous functions of two variables

$$f(x,y) = x^2+y^2-1$$
$$g(x) = \ln x$$
$$\rightarrow g(f(x,y)) = \ln(x^2+y^2-1)$$

continuous

(where it exists)

$$f(x,y) = 2x^2y+3$$
$$g(x) = e^x$$
$$\rightarrow g(f(x,y)) = e^{2x^2y+3}$$

continuous

$g(x)$ can be $\sin x$, $\cos x$, etc.

Ex. Determine the set of points on which

$$F(x,y) = \sin(xy) - \sqrt{y-x} \quad \text{is continuous.}$$

by compositions we know this is continuous everywhere it exists. So we just have to find the domain:

need $y - x \geq 0 \Rightarrow y \geq x$

$$D: \{(x,y) \in \mathbb{R}^2 : y \geq x\}.$$

Ex. Find the limit or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2}$$

we know this $= 0$ by lesson 11

We'll prove it by using <u>the Squeeze Theorem</u>:

Suppose that $|f(x,y) - L| \leq g(x,y)$ $\forall (x,y)$ in some circle centered at (a,b), except possibly at (a,b).

If $\lim\limits_{(x,y)\to(a,b)} g(x,y) = 0$, Then $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$.

This says That if $\left|\dfrac{2xy^2}{x^2+y^2}\right| \le g(x,y)$ with

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0, \text{ then } \lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = 0.$$

So let's find a <u>continuous</u> $g(x,y)$

we know limit is Then $g(0,0)$

Just plug in

$$\left|\frac{2xy^2}{x^2+y^2}\right| = \frac{2|x|y^2}{x^2+y^2} \le 2|x|$$

since $y^2 \le x^2+y^2$

let $g(x,y) = 2|x|$

continuous

so $\lim\limits_{(x,y)\to(0,0)} g(x,y) = g(0,0)$

$= 2|0| = 0$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = 0$$ by the Squeeze theorem.