Math 20300
Calculus III

Lesson 9
Vector-Valued Functions

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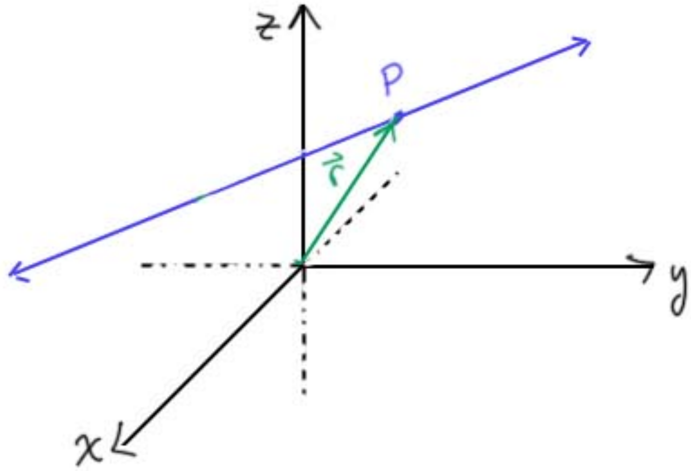
Vector Valued Functions

A <u>vector valued function</u> (or <u>vector function</u>) of $\mathbb{R}$ assigns a vector to each number in $\mathbb{R}$.

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}.$$

Ex. We've already seen the idea of a position vector:



point $P(x,y,z)$

vector $\vec{r} = \langle x,y,z \rangle$

and we know

$x = x_0 + at$

$y = y_0 + bt$

$z = z_0 + ct$

So we can think of $\vec{r}$ as a vector valued function

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

For any point on the line, $\vec{r}(t)$ is the vector from the origin to that point.

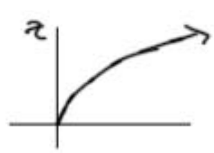
For any $t \in \mathbb{R}$, $\vec{r}(t)$ gives the vector from the origin to the point on the line } vector valued function

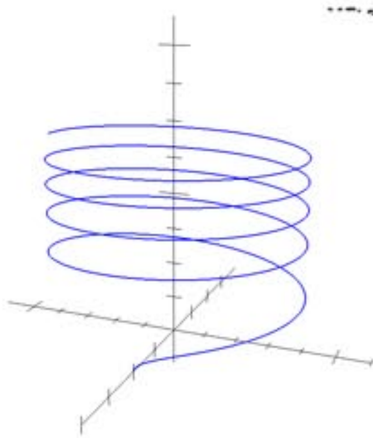
Ex. Sketch the curve defined by the vector function $\vec{r}(t) = \langle \cos t, 2\sin t, \sqrt{t} \rangle$

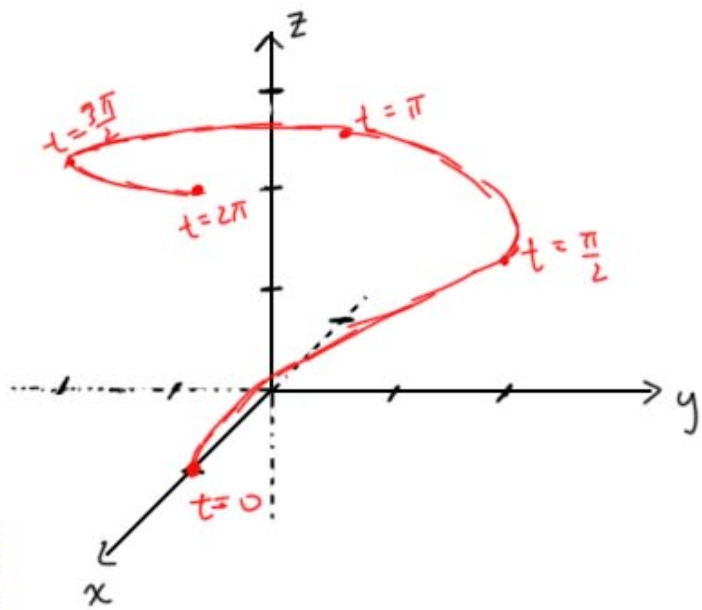
$x(t) = \cos t \in [-1,1]$

$y(t) = 2\sin t \in [-2,2]$

$z(t) = \sqrt{t} \in [0,\infty)$







$t=0$ $\quad x=1$, $y=0$, $z=0$

$t=\frac{\pi}{2}$ $\quad x=0$, $y=2$, $z=\sqrt{\frac{\pi}{2}} \approx 1.25$

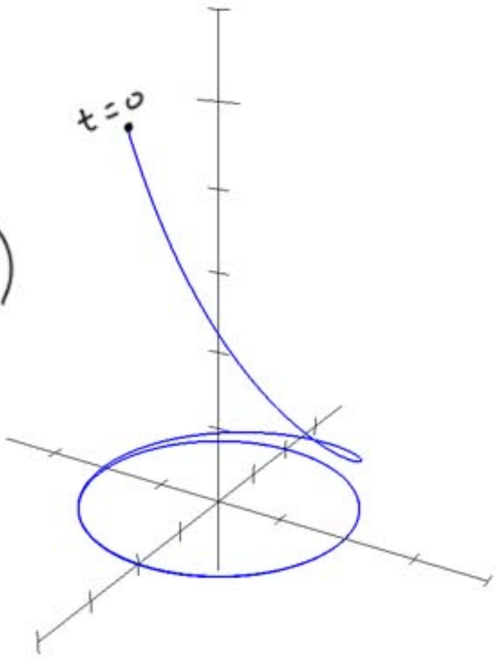
$t=\pi$ $\quad x=-1$, $y=0$, $z=\sqrt{\pi} \approx 1.77$

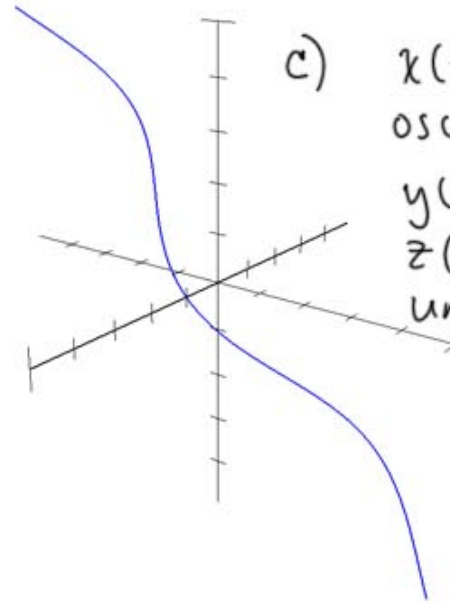
$t=\frac{3\pi}{2}$ $\quad x=0$, $y=-2$, $z=\sqrt{\frac{3\pi}{2}} \approx 2.17$

$t=2\pi$ $\quad x=1$, $y=0$, $z=\sqrt{2\pi} \approx 2.51$

Ex. Matching a vector-valued function to its graph.

b)





c) $x(t)=\cos t$
oscillating
$y(t)=-t$
$z(t)=t$
unbounded.

a) $\vec{r}(t)=\langle t\cos t, t\sin t, t\rangle$

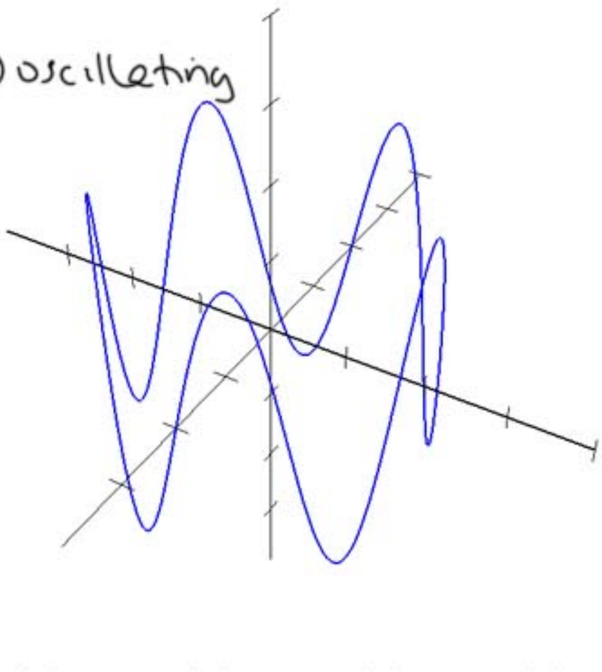
b) $\vec{r}(t)=\langle \cos t, \sin t, 5e^{-t}\rangle$

c) $\vec{r}(t)=\langle \cos t, -t, t\rangle$

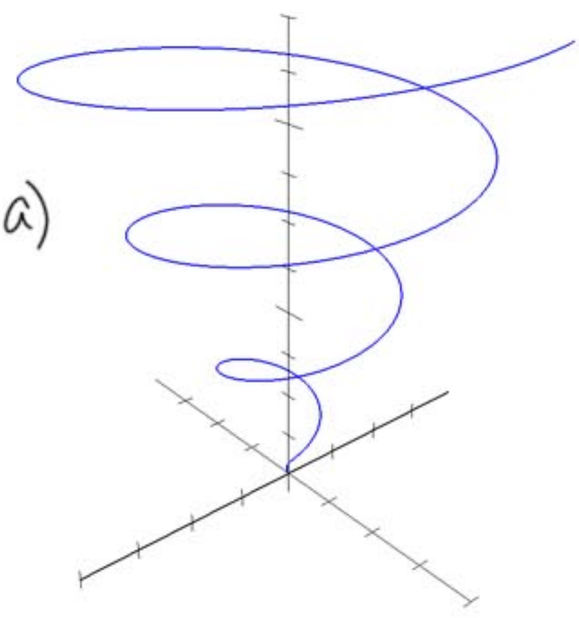
d) $\vec{r}(t)=\langle \sin t, \cos t, \sin 5t\rangle$

$z(t)$ oscillating

d)

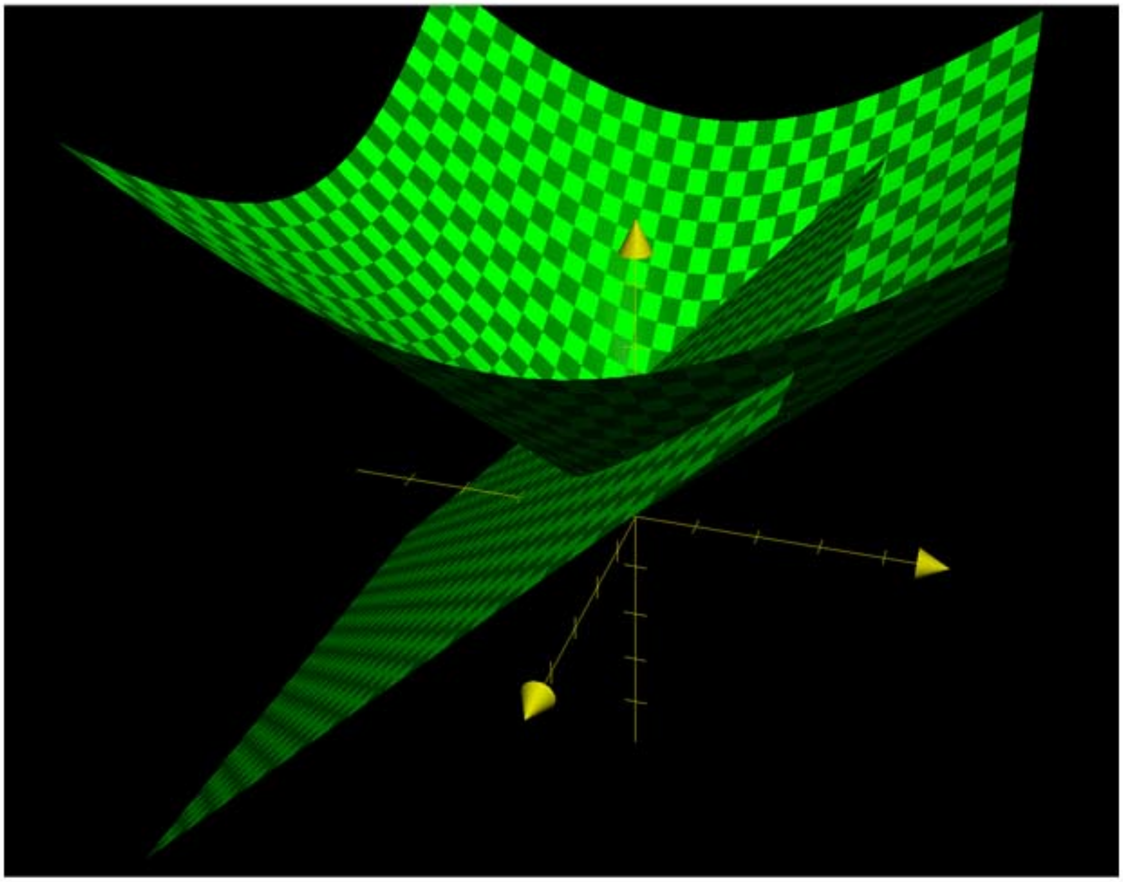


a)



Ex. Find a vector function that represents the curve of intersection of the cone $z = \sqrt{x^2+y^2}$ and the plane $y - z = -2$

from $z^2 = x^2 + y^2$ top half only.



We can see that the curve is parabola

We're looking to parametrize the curve

$x = x(t)$

$y = y(t)$ then write $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$z = z(t)$

Eliminate z : $\sqrt{x^2+y^2} = y+2$

$$x^2+y^2 = (y+2)^2 = y^2+4y+4$$

$$x^2 = 4y+4 \qquad x^2-4=4y$$

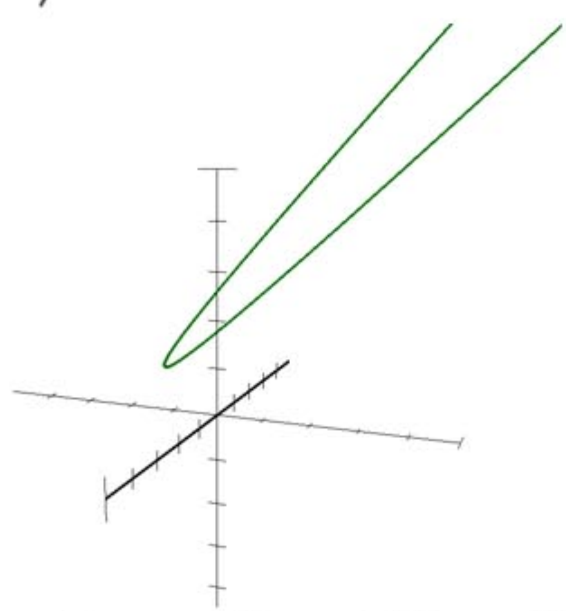
$$y = \frac{x^2}{4}-1 \qquad \text{let } x=t$$

$$y = \frac{t^2}{4}-1$$

$$z = y+2 = \frac{t^2}{4}-1+2 = \frac{t^2}{4}+1$$

$$z = \frac{t^2}{4}+1$$

so $\vec{r}(t) = \left\langle t, \frac{t^2}{4}-1, \frac{t^2}{4}+1 \right\rangle$



Limits and The Derivative

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

So that we can define

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$= \langle f'(t), g'(t), h'(t) \rangle$$

$$= \left\langle \frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt} \right\rangle$$

Then for a vector function describing a curve in $\mathbb{R}^3$,

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\vec{r}\,'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

rate of change of x with respect to t

and $\vec{r}\,'(t) = \left\langle \left.\frac{dx}{dt}\right|_{t_0}, \left.\frac{dy}{dt}\right|_{t_0}, \left.\frac{dz}{dt}\right|_{t_0} \right\rangle$

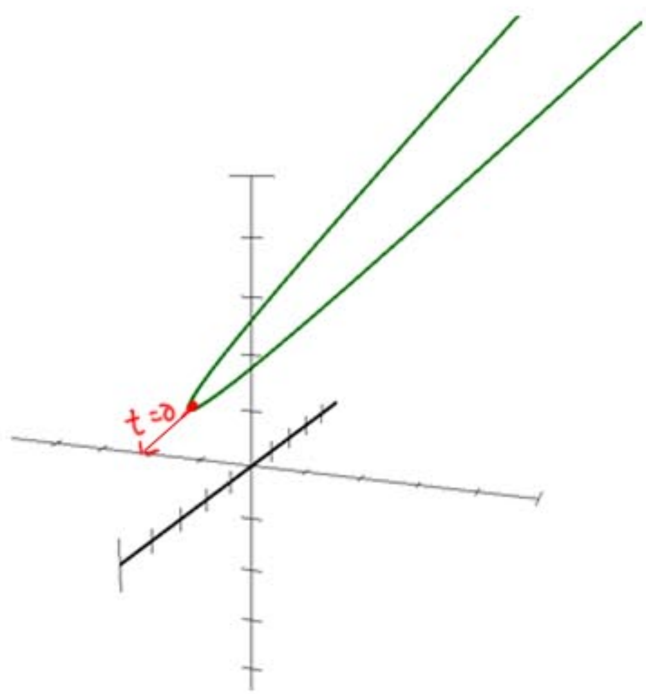
is a tangent vector to the curve

at $t = t_0$.

For the parabola in $\mathbb{R}^3$ above

$\vec{r}(t) = \left\langle t, \frac{t^2}{4} - 1, \frac{t^2}{4} + 1 \right\rangle \qquad \vec{r}(0) = \langle 0, -1, 1 \rangle$

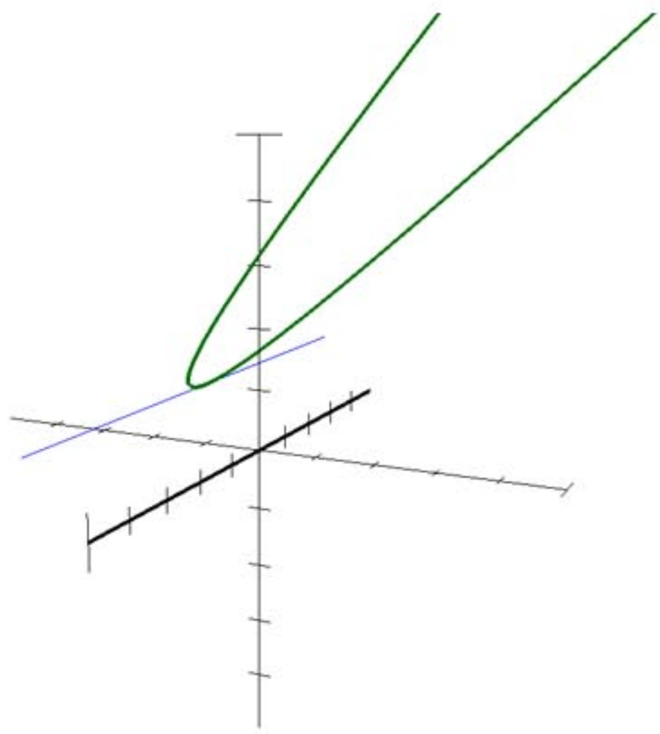
$\vec{r}\,'(t) = \left\langle 1, \frac{t}{2}, \frac{t}{2} \right\rangle \qquad \vec{r}\,'(0) = \langle 1, 0, 0 \rangle$



Now, find parametric equations for the tangent line to the curve of $\vec{r}(t)$ at $t=0$.

point $(0,-1,1)$ direction $\langle 1,0,0\rangle$

$$\begin{aligned} x &= 0+t \\ y &= -1+0t \\ z &= 1+0t \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= t \\ y &= -1 \\ z &= 1 \end{aligned}$$



Ex. For the curve of vector function $\vec{r}(t) = \langle \cos t, 2\sin t, \sqrt{t} \rangle$, find a unit tangent vector at $t = \frac{\pi}{2}$

$$\vec{r}\,'(t) = \left\langle -\sin t,\ 2\cos t,\ \frac{1}{2\sqrt{t}} \right\rangle$$

$$\vec{r}\,'\left(\tfrac{\pi}{2}\right) = \left\langle -\sin\tfrac{\pi}{2},\ 2\cos\tfrac{\pi}{2},\ \frac{1}{2\sqrt{\frac{\pi}{2}}} \right\rangle$$

$$= \left\langle -1,\ 0,\ \frac{1}{\sqrt{2\pi}} \right\rangle$$

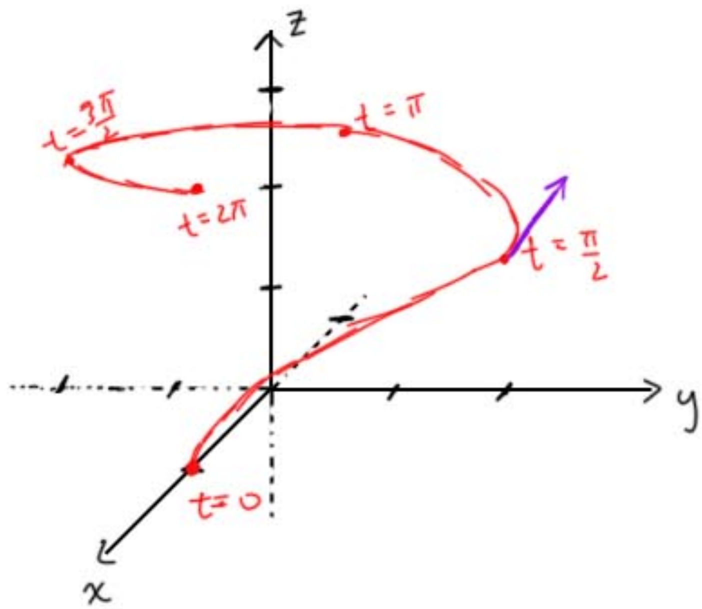
tangent vector

$$\frac{1}{2\sqrt{\frac{\pi}{2}}}\,\frac{\sqrt{\frac{\pi}{2}}}{\sqrt{\frac{\pi}{2}}} = \frac{\sqrt{\frac{\pi}{2}}}{\pi} = \frac{1}{\pi}\cdot\frac{\sqrt{\pi}}{\sqrt{2}} = \frac{1}{\sqrt{2\pi}}$$

$$\left\|\vec{r}\,'\left(\tfrac{\pi}{2}\right)\right\| = \sqrt{1 + \tfrac{1}{2\pi}}$$

$$\vec{u} = \frac{1}{\sqrt{1+\frac{1}{2\pi}}}\left\langle -1,\ 0,\ \frac{1}{\sqrt{2\pi}} \right\rangle$$

$$= \left\langle \frac{-1}{\sqrt{1+\frac{1}{2\pi}}},\ 0,\ \frac{1}{\sqrt{2\pi+1}} \right\rangle.$$



z
y
x
t = π
t = 3π/2
t = 2π
t = π/2
t = 0