Math 20300
Calculus III

Lesson 5
Lines in Three Dimensional Space

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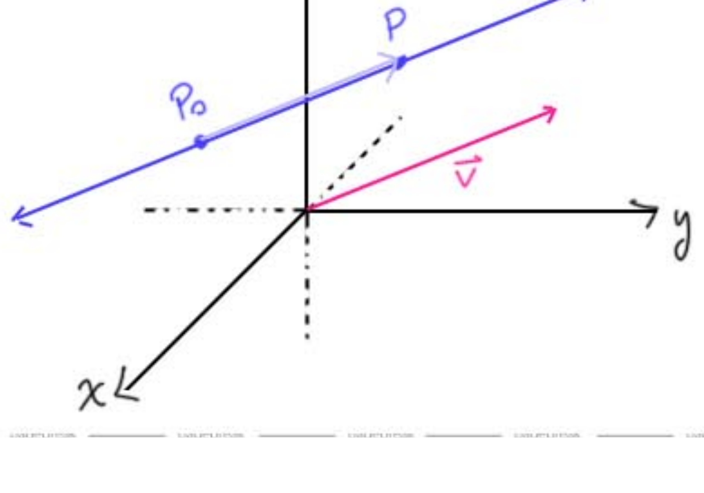
Lines in Three Dimensional Space.

In $\mathbb{R}^2$, a line is specified by a point it goes through, and its slope. Sometimes we are given two points, and we find the slope.

In $\mathbb{R}^3$, a line is specified by a point it goes through, and its direction (given by a vector). We can also find equation(s) of the line by using two points on the line, and finding a direction vector.



Given $P_0(x_0, y_0, z_0)$ on the line and

direction vector $\vec{v} = \langle a, b, c \rangle$,

for any point $P(x, y, z)$ on the line,

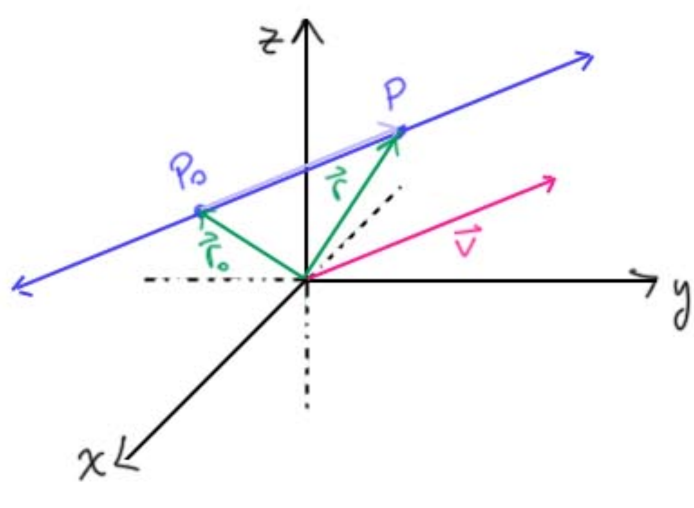
we have $\underbrace{\overrightarrow{P_0P}}_{\substack{\text{vector} \\ \text{determined} \\ \text{by } P_0 \,\&\, P}}$ must be parallel to $\underbrace{\vec{v}}_{\substack{\text{direction} \\ \text{vector} \\ \text{for the line}}}$

$\therefore \overrightarrow{P_0P} = t\vec{v}$ for some scalar t

$$\langle x - x_0, y - y_0, z - z_0 \rangle = t\langle a, b, c \rangle$$
$$= \langle at, bt, ct \rangle$$

$$\therefore \begin{aligned} x - x_0 &= at \\ y - y_0 &= bt \\ z - z_0 &= ct \end{aligned} \Rightarrow \boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned} \quad \begin{array}{l} \text{Parametric} \\ \text{equations} \\ \text{for the line} \\ \text{through } (x_0, y_0, z_0) \\ \text{in direction } \langle a, b, c \rangle. \end{array}}$$

In vector form,



position vectors
$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$
$\vec{r} = \langle x, y, z \rangle$

then $\vec{r} = \vec{r}_0 + \underbrace{\overrightarrow{P_0P}}$

$\vec{r} = \vec{r}_0 + t\vec{v}$ vector equation of the line.

Ex. Find parametric equations and a vector equation for the line through $(1,-2,5)$ parallel to (in the direction of) $\langle 3,-1,2 \rangle$.

$x = x_0 + at \qquad x = 1 + 3t$

$y = y_0 + bt \qquad y = -2 - t$ parametric equations

$z = z_0 + ct \qquad z = 5 + 2t$

for the vector equation, need a position vector

for the given point $(1,-2,5)$ $\quad \vec{r}_0 = \langle 1, -2, 5 \rangle$

and direction vector $\vec{v} = \langle 3, -1, 2 \rangle$

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

$$\vec{r} = \vec{i} - 2\vec{j} + 5\vec{k} + t\left(3\vec{i} - \vec{j} + 2\vec{k}\right)$$

$$= (1+3t)\vec{i} - (2+t)\vec{j} + (5+2t)\vec{k} \, .$$

Notice we found

$x = 1 + 3t$
$y = -2 - t$
$z = 5 + 2t$

parametric equations

from direction vector $\vec{v} = \langle 3, -1, 2 \rangle$.

Consider vector $2\vec{v} = \langle 6, -2, 4 \rangle$ same direction!

through same point $(1,-2,5)$ must be same line

$x = 1 + 6t$ same set of points,

$y = -2 - 2t$ but different parametrization.

$z = 5 + 4t$ both have point $(1,-2,5)$ at $t = 0$

but the first parametrization moves slower, ie it takes $t=2$ to get to point $(7,-4,9)$ where in the second parametrization, $t=1$ gives $(7,-4,9)$.

Ex. Find parametric equations of the line through the points $P(2,1,3)$ and $Q(4,0,2)$.

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

Parametric equations for the line through (x_0, y_0, z_0) in direction $\langle a, b, c\rangle$.

use

$\vec{v} = \overrightarrow{PQ} = \langle 2,-1,-1\rangle$

or

$\vec{v} = \overrightarrow{QP} = \langle -2,1,1\rangle$

$P+\overrightarrow{PQ}$

$$x = 2+2t$$
$$y = 1-t$$
$$z = 3-t$$

$P+\overrightarrow{QP}$

$$x = 2-2t$$
$$y = 1+t$$
$$z = 3+t$$

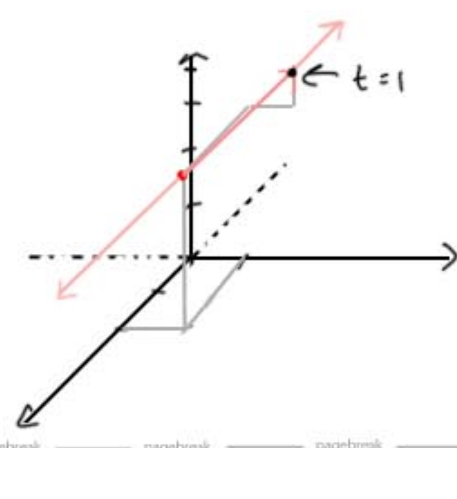
$Q+\overrightarrow{PQ}$

$$x = 4+2t$$
$$y = -t$$
$$z = 2-t$$

$Q+\overrightarrow{QP}$

$$x = 4-2t$$
$$y = t$$
$$z = 2+t$$

parametrizations will be at same speed, but in different directions



Parallel, Intersecting, and Skew lines.

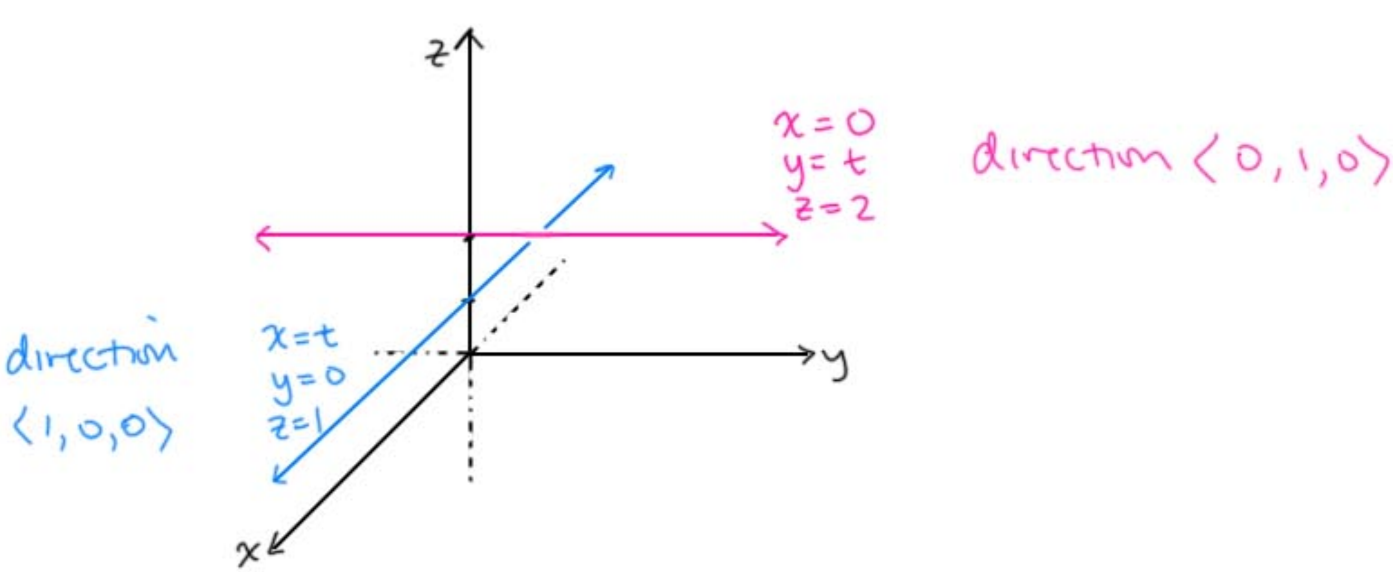
Lines are <u>parallel</u> if they have the same direction, ie if their direction vectors are parallel.

Ex. $x = 2 + 3t$, $y = -1 + 2t$, $z = 1 - t$ and $x = 5 - 9t$, $y = 1 - 6t$, $z = 2 + 3t$ are parallel

$\langle 3, 2, -1 \rangle$ $\qquad$ $\langle -9, -6, 3 \rangle$

In $\mathbb{R}^2$, if lines are not parallel, they intersect.

In $\mathbb{R}^3$, not so.



Non-intersecting, non-parallel lines are called <u>skew lines</u>.

Ex. Are these lines ~~parallel~~, intersecting, or skew?

ℓ_1: $x = 4 + t$, $y = 2$, $z = 3 + 2t$ $\quad\langle 1, 0, 2\rangle$

ℓ_2: $x = 2 + 2s$, $y = 2s$, $z = -1 + 4s$ $\quad\langle 2, 2, 4\rangle$

Is there a pair (t, s) that gives the same point (x, y, z) on both lines?

if so,

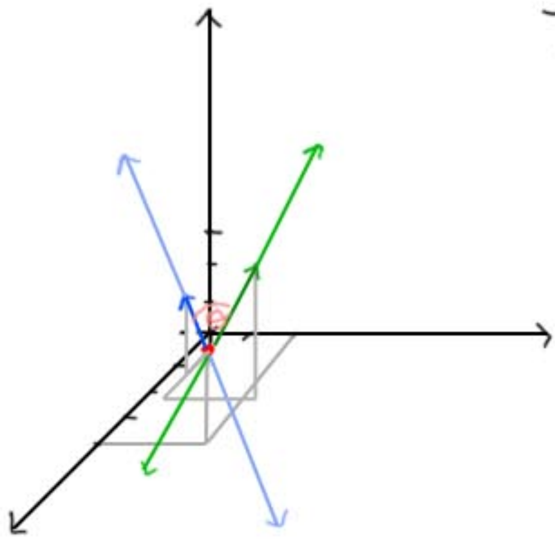
$$4 + t = 2 + 2s \qquad 2 = 2s \qquad 3 + 2t = -1 + 4s$$

$$4 + t = 4 \qquad s = 1 \qquad 3 + 0 \stackrel{?}{=} -1 + 4 \ \checkmark$$

$$t = 0$$

these lines intersect when $t = 0$ (ℓ_1) and $s = 1$ (ℓ_2)

ie, at point $(4, 2, 3)$



The angle between intersecting lines is the angle between their direction vectors

$$\ell_1 : \langle 1,0,2 \rangle \quad \ell_2 : \langle 2,2,4 \rangle$$
$$\vec{a} \qquad\qquad \vec{b}$$

$$\|\vec{a}\| = \sqrt{1^2+0^2+2^2} = \sqrt{5}$$

$$\|\vec{b}\| = \sqrt{2^2+2^2+4^2} = \sqrt{24} = 2\sqrt{6}$$

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$2+0+8 = \sqrt{5}\,2\sqrt{6}\cos\theta$$

$$\cos\theta = \frac{10}{2\sqrt{30}} \qquad \theta \approx .42 \text{ rad} \approx 24^\circ$$

If we eliminate the parameter in the equations for a line, we get

$$x = x_0 + at \Rightarrow t = \frac{x-x_0}{a}$$

$$y = y_0 + bt \Rightarrow t = \frac{y-y_0}{b} \qquad a,b,c \neq 0$$

$$z = z_0 + ct \quad \Rightarrow \quad t = \frac{z - z_0}{c}$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

symmetric equations of the line through (x_0, y_0, z_0) in direction $\langle a, b, c \rangle$

Ex. l_2 above

$$x = 2 + 2s$$
$$y = 2s$$
$$z = -1 + 4s$$

$$\frac{x-2}{2} = \frac{y}{2} = \frac{z+1}{4}$$

Even if $a, b,$ or $c = 0$, can still eliminate parameter:

Ex. l_1 above

$$x = 4 + t$$
$$y = 2$$
$$z = 3 + 2t$$

$$y = 2, \quad x - 4 = \frac{z-3}{2}$$