Math 20200
Calculus II

Lesson 27
Conic Sections

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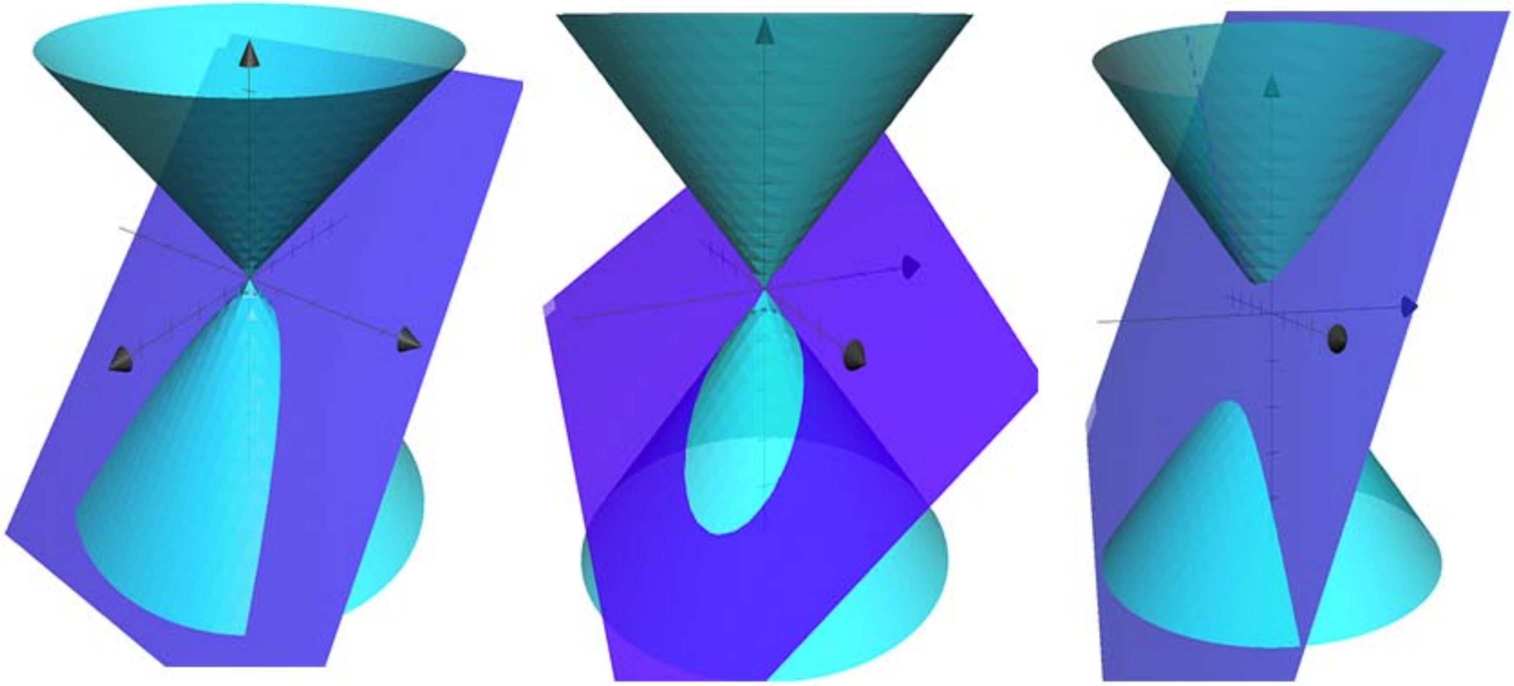
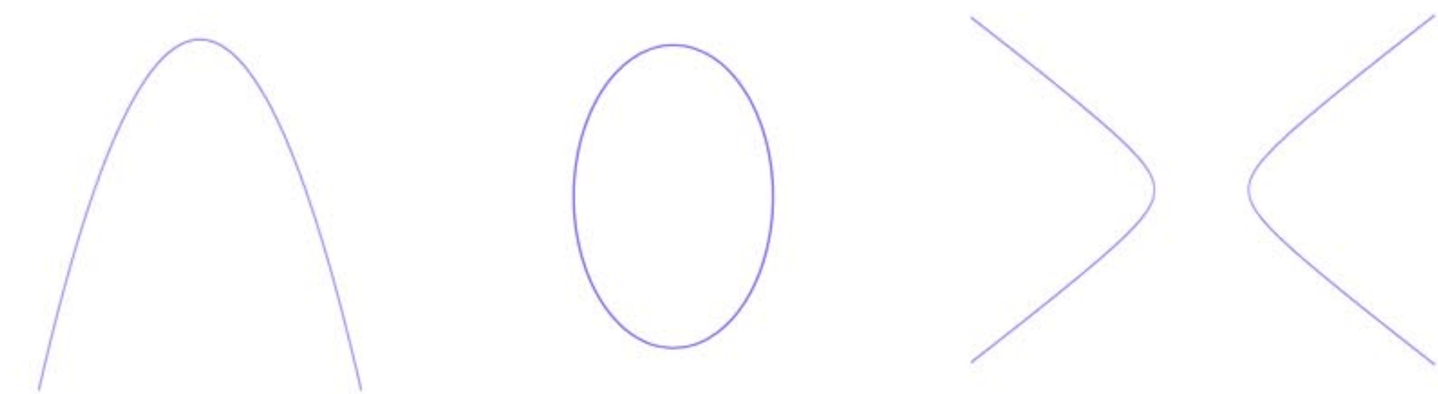
Conic Sections

Consider the curve of intersection of a cone and a plane:

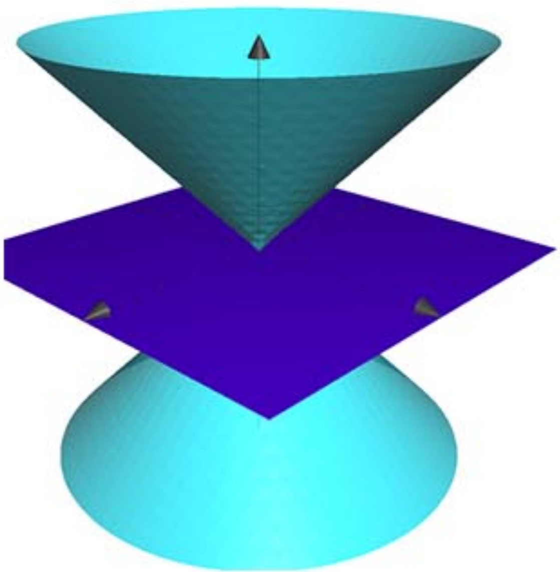


parabola ellipse hyperbola

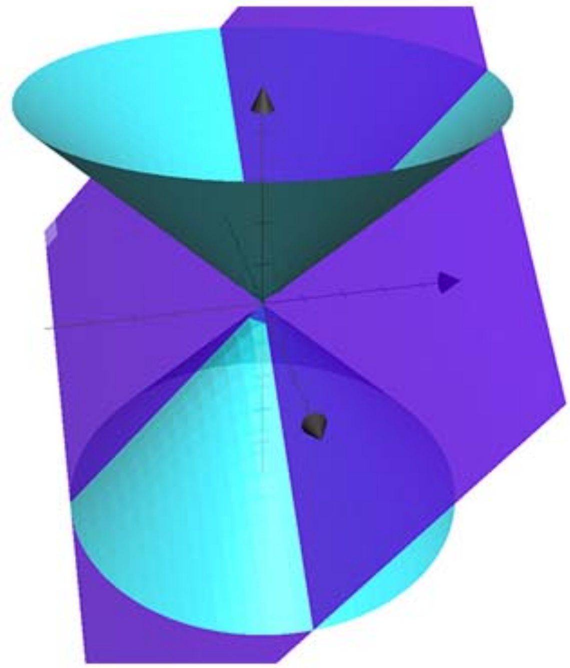


These curves are called conic sections.

Note the degenerate cases when the plane goes through the vertex of the cone :



intersection is just one point

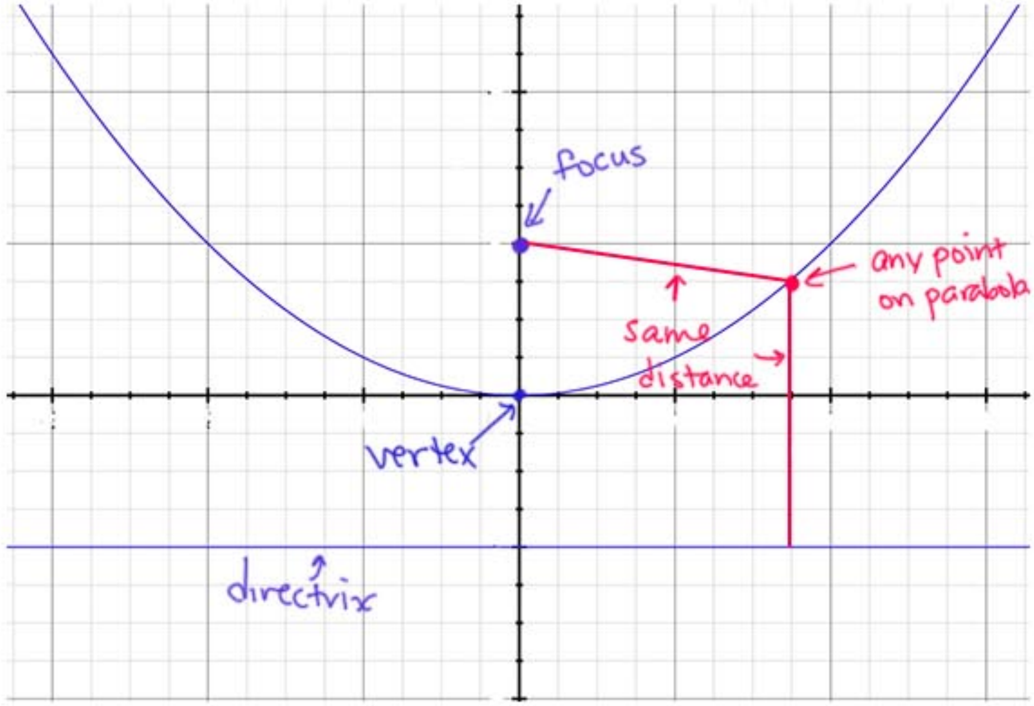


intersection is a pair of lines

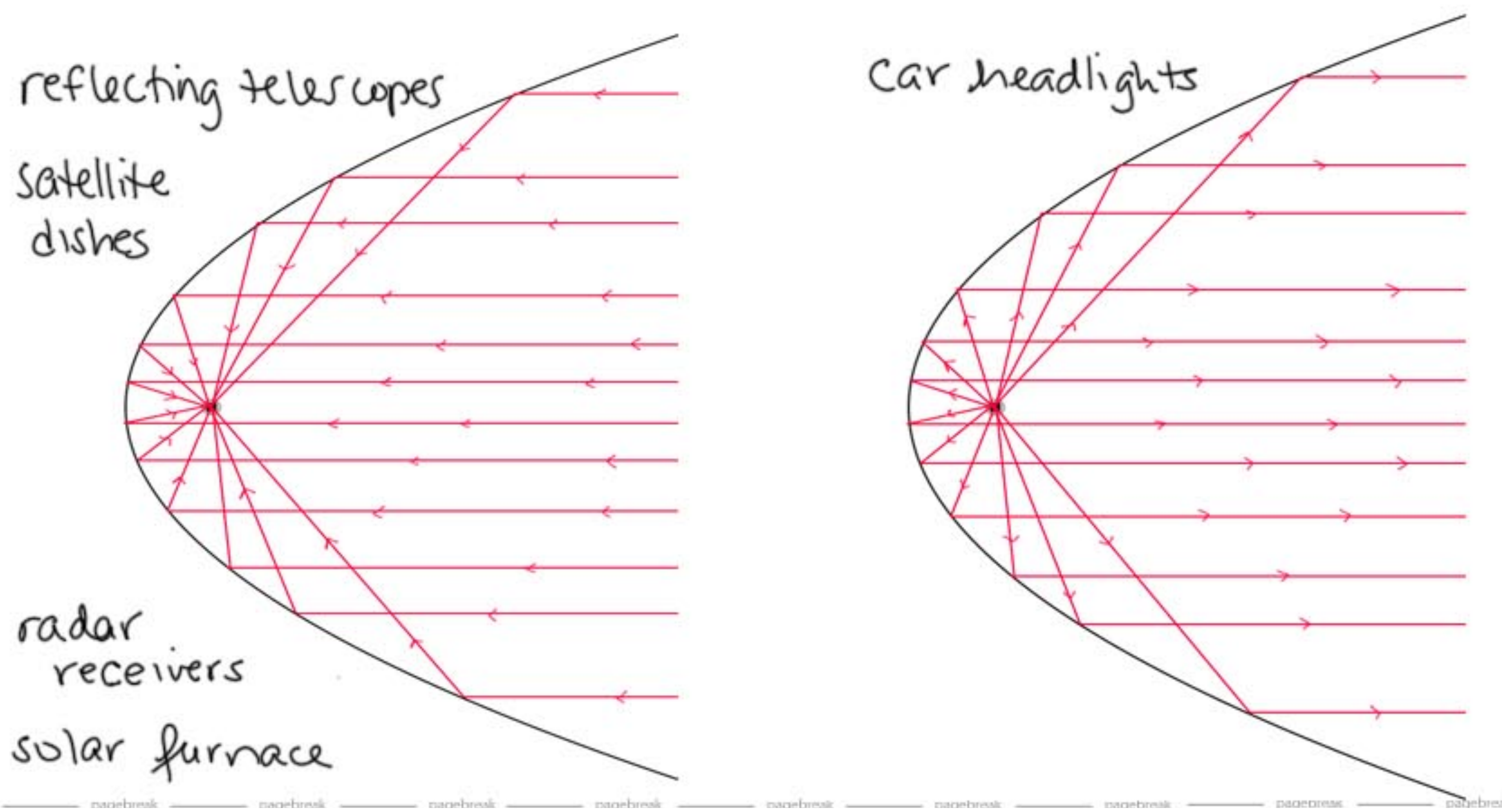
We are not interested in these cases.

Parabolas :

Definition: A <u>parabola</u> is the set of points in the plane equidistant from a given focus point and a given directrix line.



Applications of the parabola (or the paraboloid, 3D version):



Parabolic trajectories:

motion in 2 dimensions:

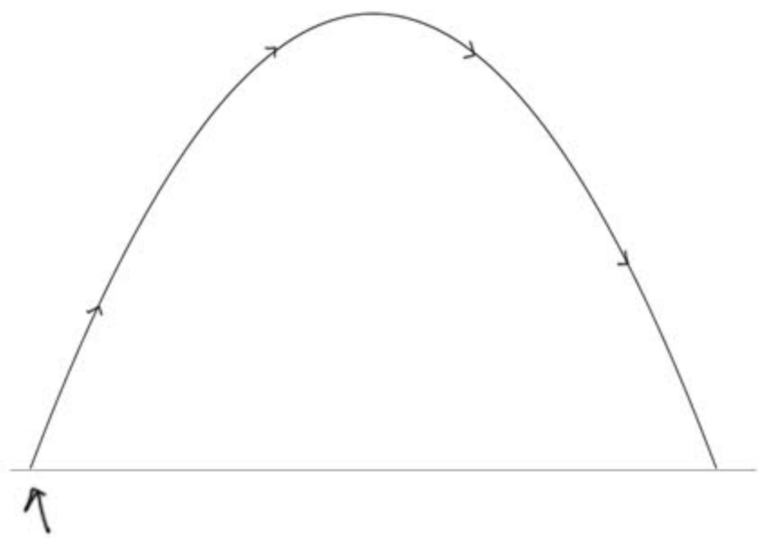
hitting a baseball

punting a football

shooting a basketball

water fountain

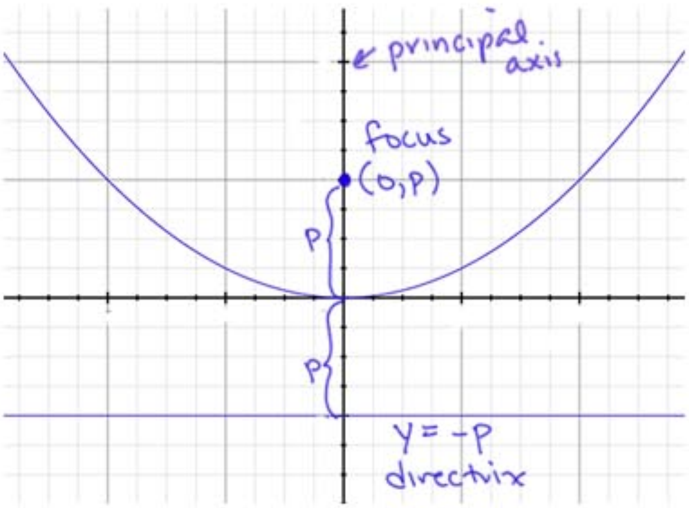
angry birds



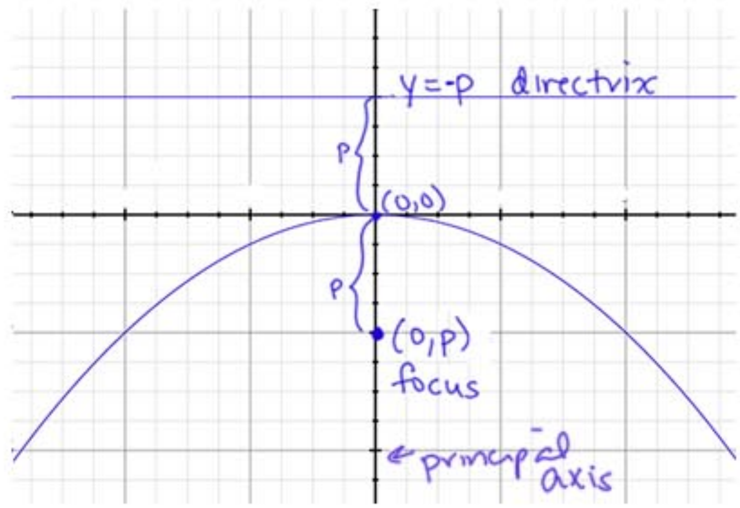
starting with an initial velocity and the angle of incline, we can predict how far the object will travel and how long it will take.
(near the earth's surface, neglecting air resistance)

Also, NASA's KC-135 "Vomit Comet" aircraft follows parabolic paths to create the zero-gravity effect.

When a parabola is centered at the origin :



$p>0$



$p<0$

Setting the red distances (above) equal gives :

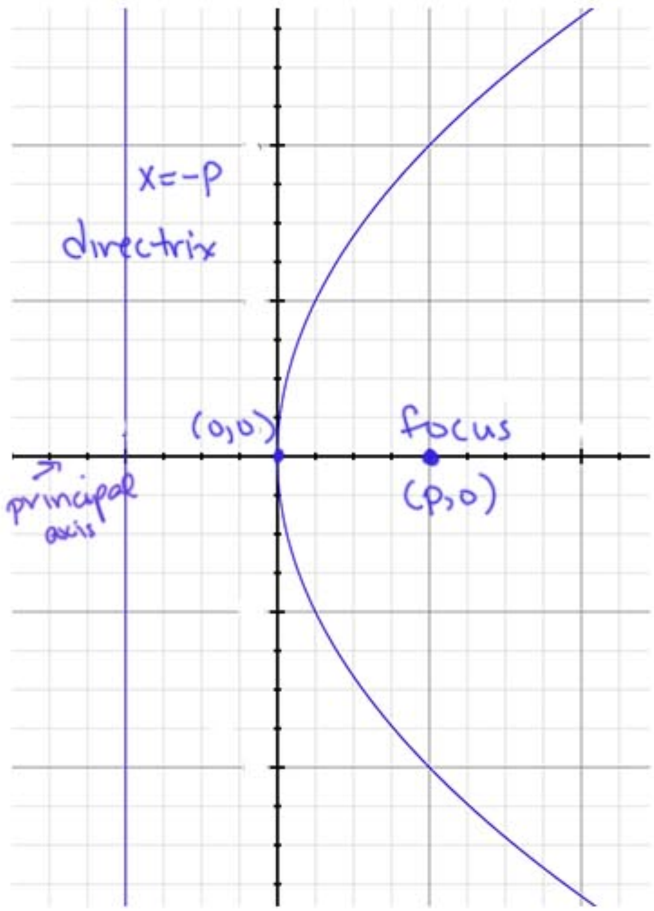
$$x^2 = 4py.$$

Ex. Find an equation of the parabola with focus (0, 3) and directrix $y = -3$.

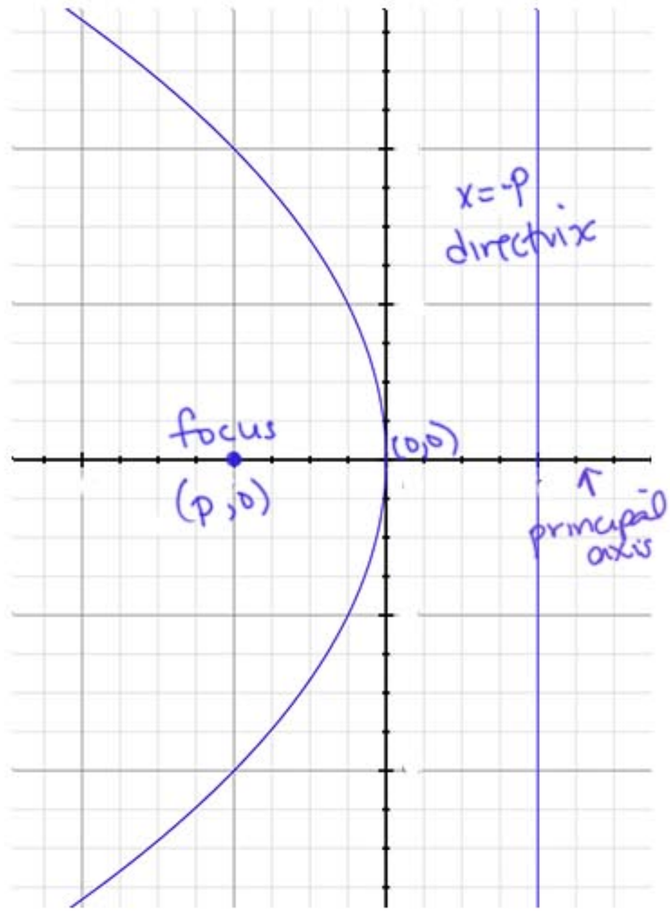
$$\Rightarrow \; p = 3 \quad \therefore \quad x^2 = 4(3)y = 12y$$

$$x^2 = 12y \quad \text{or} \quad y = \tfrac{1}{12}x^2.$$

Similarly,



$p>0$



$p<0$

and $\quad y^2 = 4px$.

Ex. Find an equation of the parabola with focus $(-4,0)$ and vertex $(0,0)$.

$$\Rightarrow p = -4 \quad \therefore \quad y^2 = 4(-4)x = -16x$$

$$y^2 = -16x \quad \text{or} \quad x = -\frac{1}{16}y^2.$$

Ex. Find the focus and directrix of the parabola

$$y = -2x^2.$$

we know this must fit $\quad x^2 = 4py$

$$y = \frac{1}{4p} x^2$$

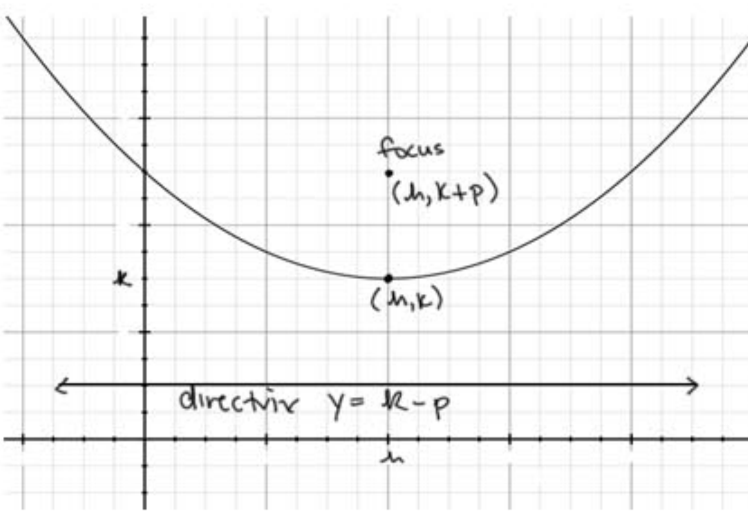
$\therefore \quad -2 = \frac{1}{4p}$ and $p = -\frac{1}{8}$. (downward facing)

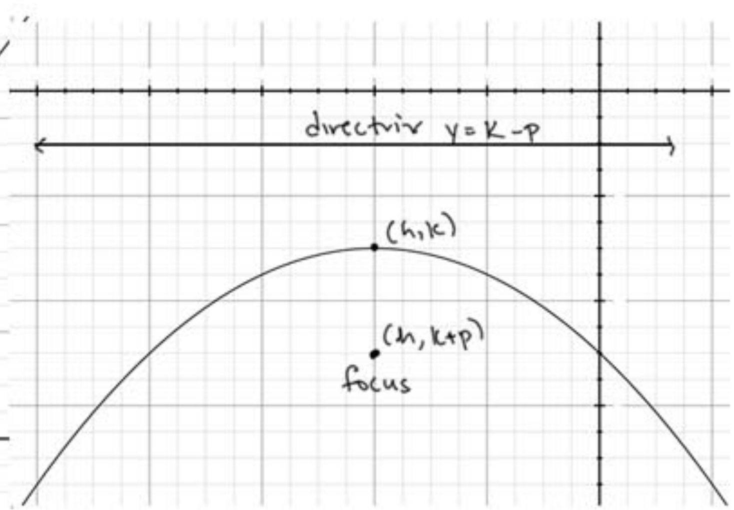
focus $(0, p) = (0, -\frac{1}{8})$ and directrix $y = -p$
$y = \frac{1}{8}$.

Shifted parabolas:

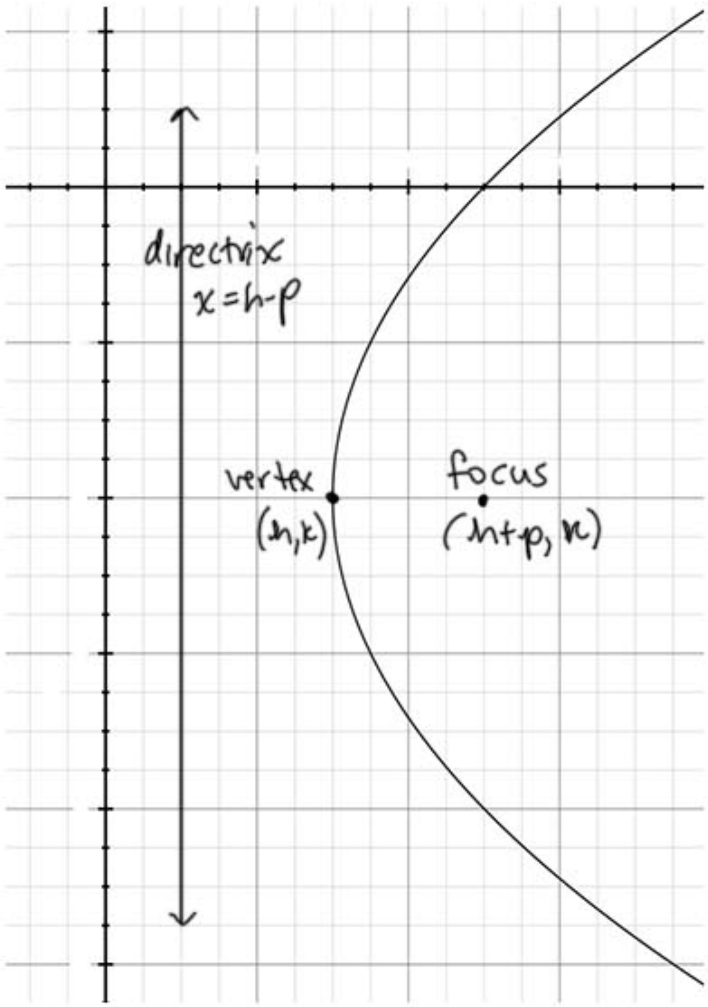
for a parabola with vertex (h,k), replace x with $x-h$ and y with $y-k$:

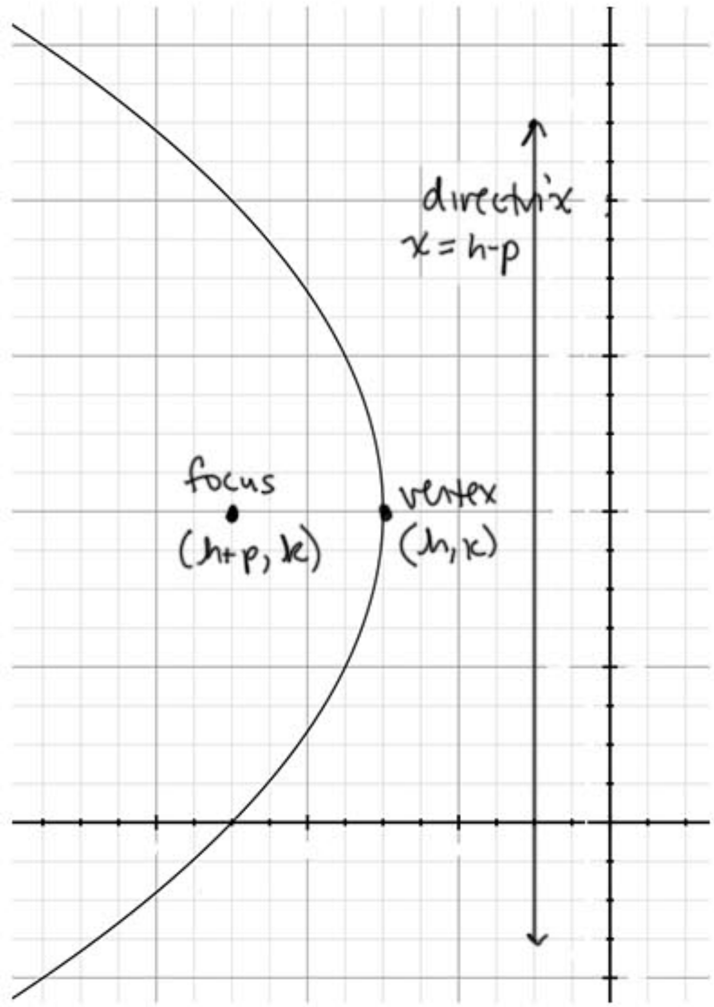
$$(x-h)^2 = 4p(y-k):$$



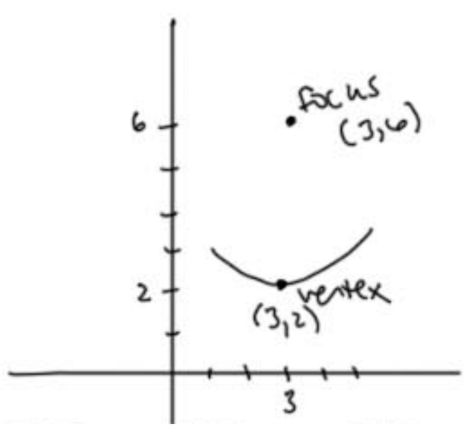


and $(y-k)^2 = 4p(x-h)$:





Ex. Find an equation of the parabola with focus $(3,6)$ and vertex $(3,2)$.



we know $(x-h)^2 = 4p(y-k)$

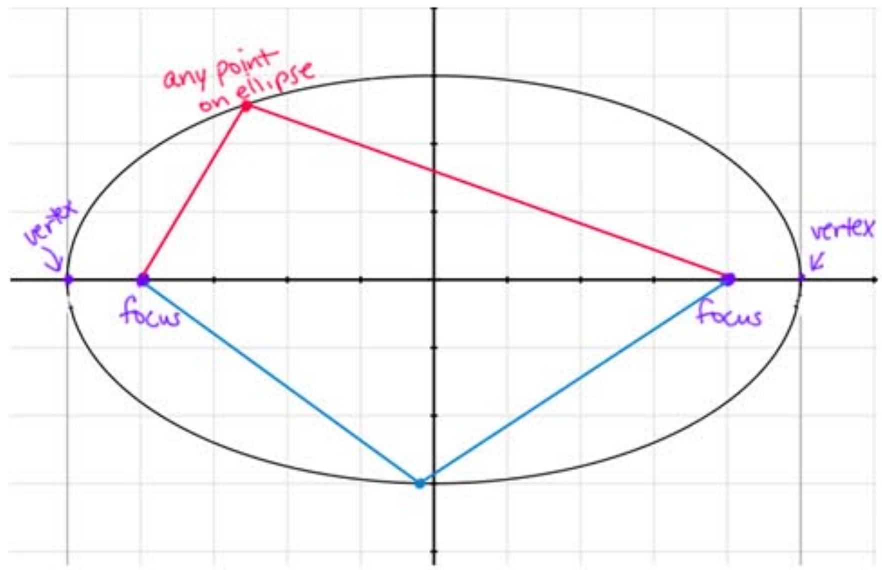
opening upward $\Rightarrow p>0$

$p=4$

$\therefore (x-3)^2 = 16(y-2)$.

Ellipses:

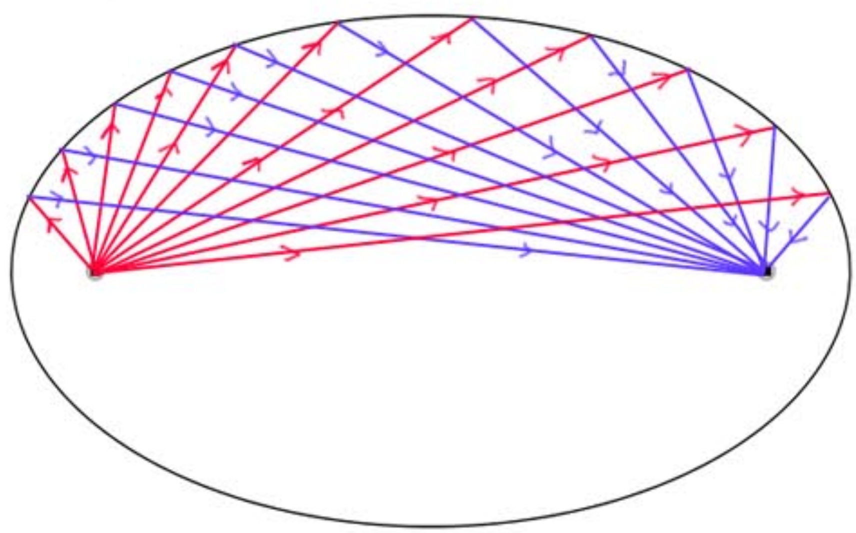
Definition: An <u>ellipse</u> is the set of points for which the sum of the distances to two focus points is constant.



Applications of The ellipse (or ellipsoid, 3D version):

lithotripter - breaks down kidney stones

whispering galleries

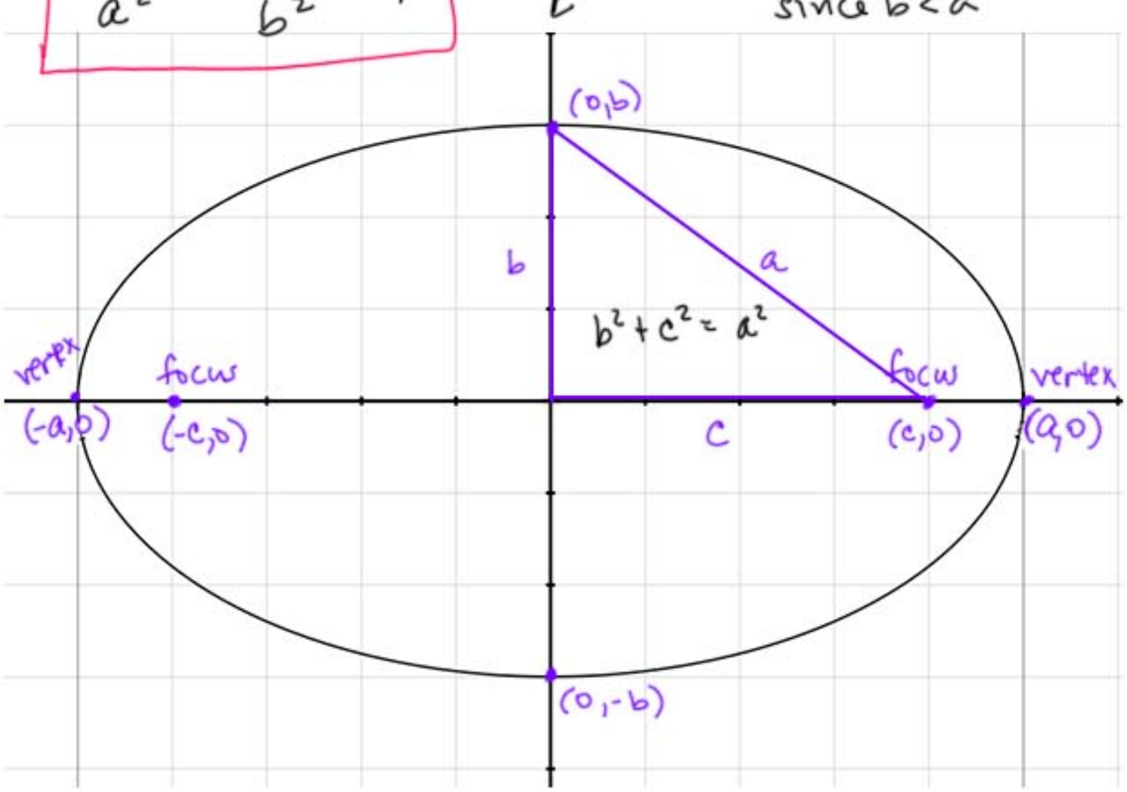


Also, planets orbit The sun over an elliptical path, sun's center of mass at one focus.

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

minor axis since $b < a$

(Centered at the origin)

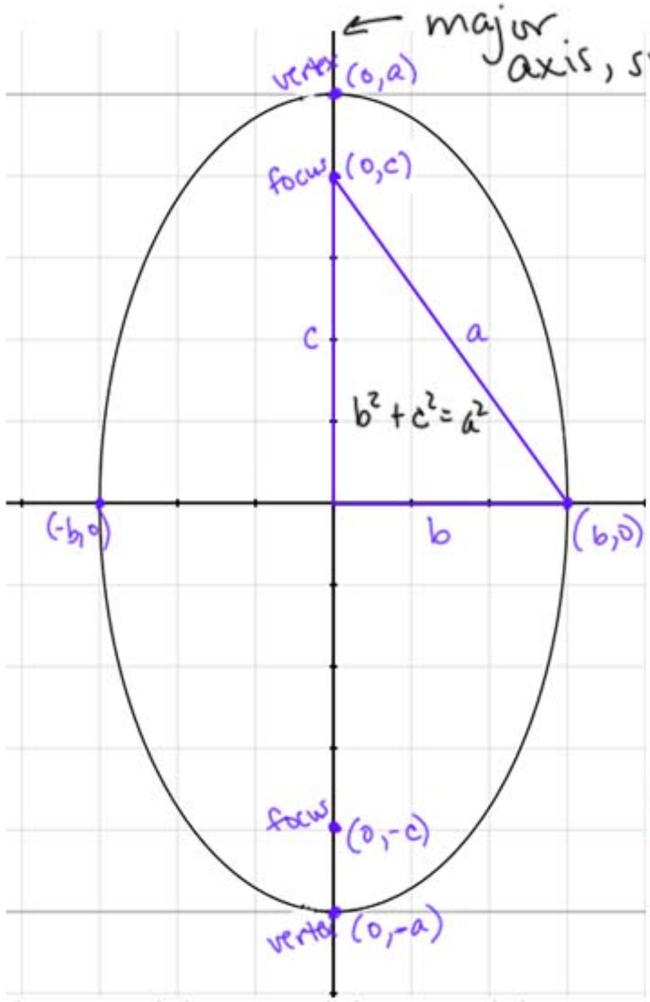


← major axis since $a > b$

OR

← major axis, since $a > b$

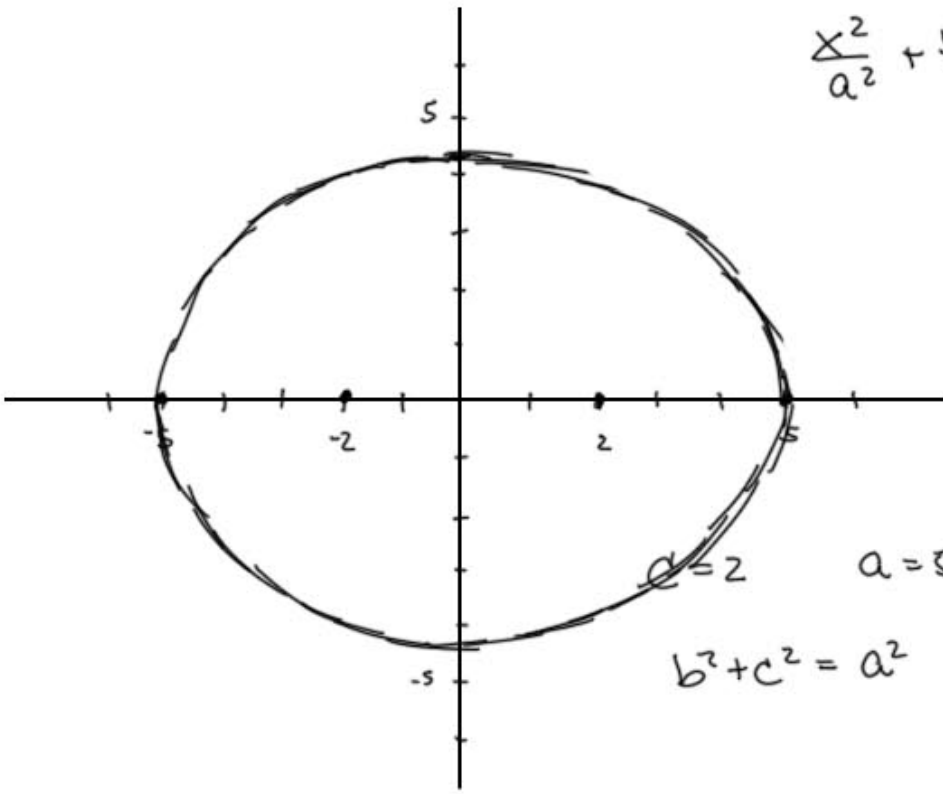
(also centered at The origin)



← minor axis since $b < a$

$$\boxed{\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1}$$

Ex. Find an equation for the ellipse with foci $(\pm 2, 0)$ and vertices $(\pm 5, 0)$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$c = 2 \qquad a = 5$

$b^2 + c^2 = a^2$

we know $a = 5$, $c = 2$. What is b?

$$b^2 + 2^2 = 5^2$$

$$b^2 + 4 = 25$$

$$b^2 = 21 \Rightarrow b = \sqrt{21}$$

note $\sqrt{16} < \sqrt{21} < \sqrt{25}$
$4 < \sqrt{21} < 5$

$$\therefore \quad \frac{x^2}{25} + \frac{y^2}{21} = 1$$

Ex. Find the vertices and foci of the ellipse and sketch the graph:

$$9x^2 + 4y^2 = 36.$$

Work on this problem on your own

we need a 1 on the right hand side, so divide by 36:

$$\frac{9x^2}{36} + \frac{4y^2}{36} = \frac{36}{36}$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \qquad \therefore a = 3, \quad b = 2$$

major axis is y axis

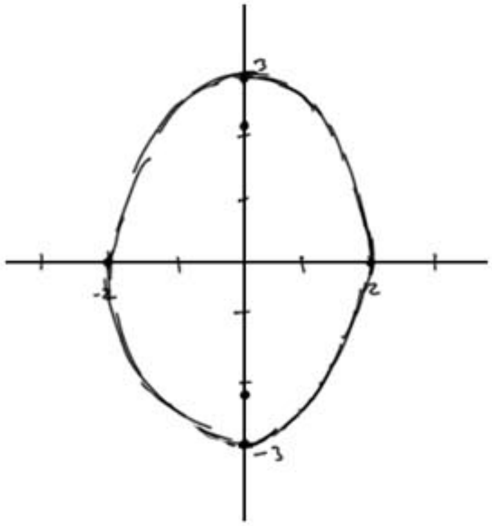
$$2^2 + c^2 = 3^2$$

$$4 + c^2 = 9$$

$$c^2 = 5 \qquad c = \sqrt{5}$$

foci $(0, \pm\sqrt{5})$ vertices $(0, \pm 3)$

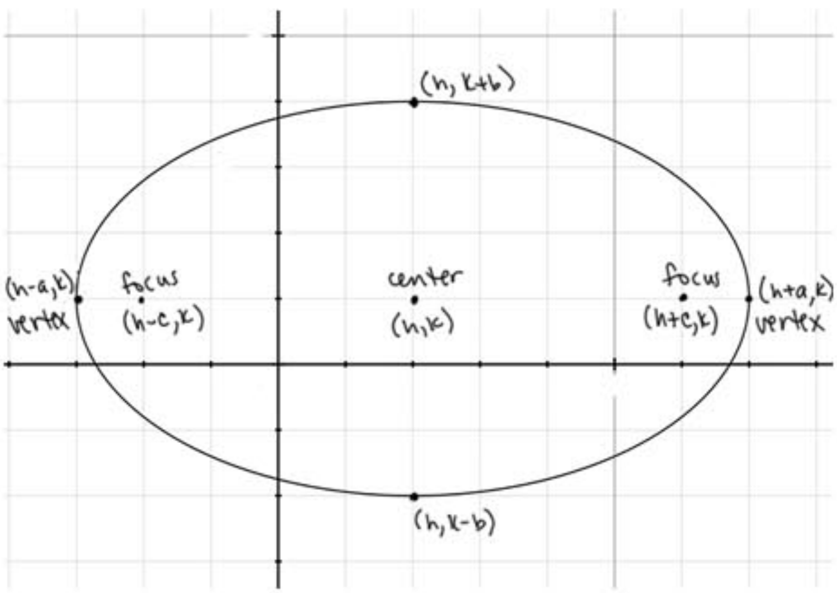
sketch:



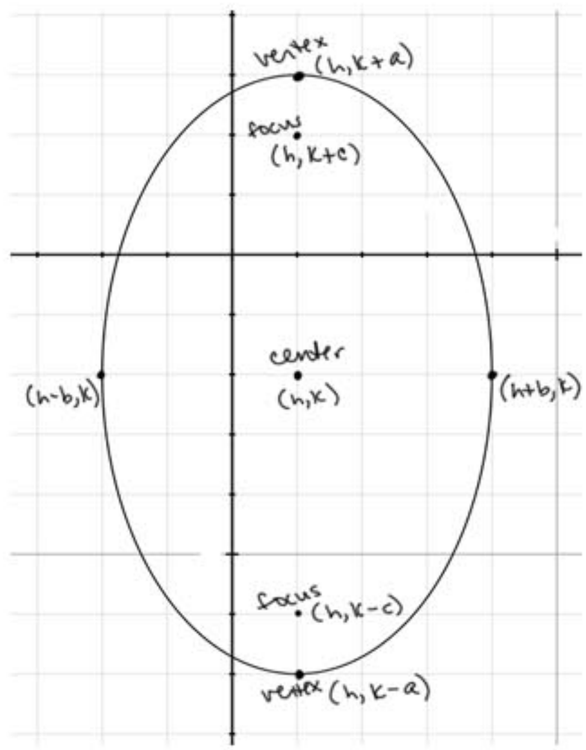
Shifted ellipses:

for an ellipse centered at (h,k),

replace x with $x-h$ and y with $y-k$.



$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

Ex. Find the vertices and foci of the ellipse and sketch the graph:

$$x^2+2y^2-6x+4y+7=0$$

complete the squares:

$$x^2-6x \quad +2y^2+4y \quad +7=0$$

$$x^2-6x \quad +2\left[y^2+2y\right]+7=0$$

$$(x-3)^2-9 \quad +2\left[(y+1)^2-1\right]+7=0$$

$x^2-6x+9-9$
$=x^2-6x$

$y^2+2y+1-1$
$=y^2+2y$

$$(x-3)^2-9+2(y+1)^2-2+7=0$$

$$(x-3)^2+2(y+1)^2=4 \qquad \text{divide by } 4$$

$$\frac{(x-3)^2}{4}+\frac{(y+1)^2}{2}=1$$

$\therefore\ a=2 \quad b=\sqrt{2}$

$b^2+c^2=a^2$

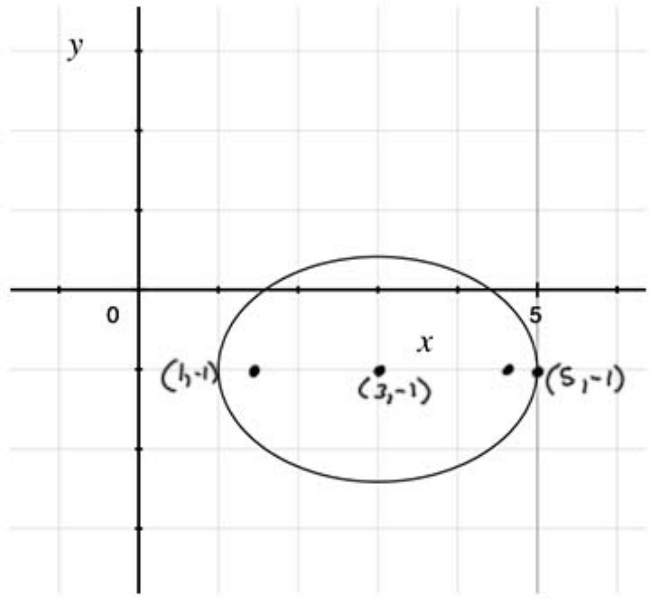
$2+c^2=4$

$c^2=2 \quad c=\sqrt{2}$

center is $(3,-1)$ vertices add + subtract $a=2$
from the x value of 3
(since x is the major axis)

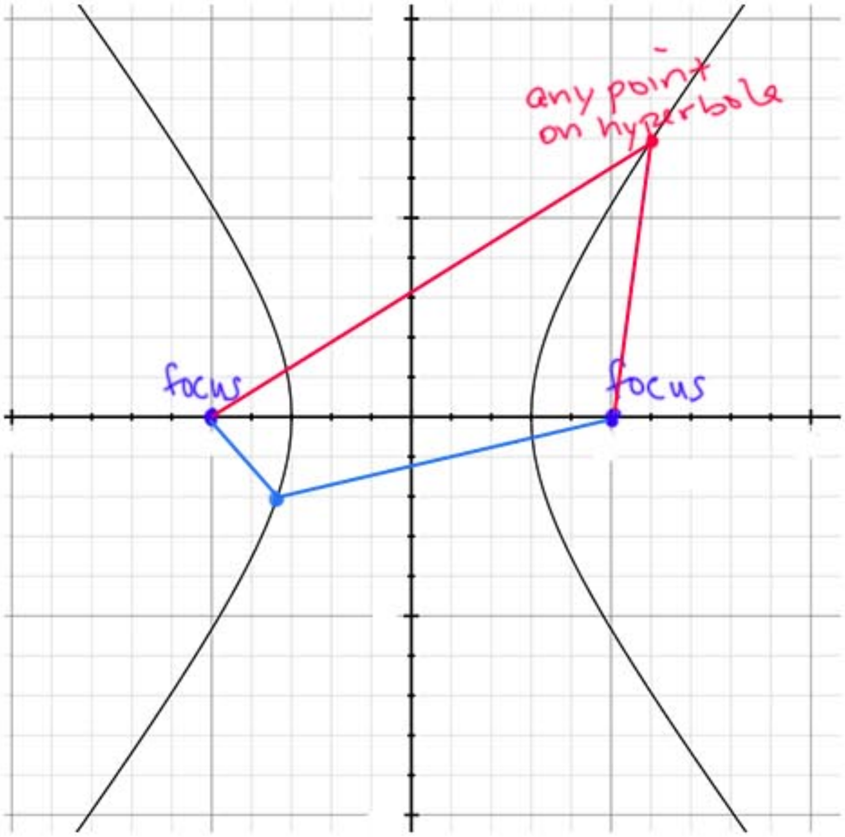
$\therefore$ vertices $(3 \pm 2, -1) \Rightarrow (5,-1)$ and
$(1,-1)$

and foci $(3 \pm \sqrt{2}, -1)$.



Hyperbolas:

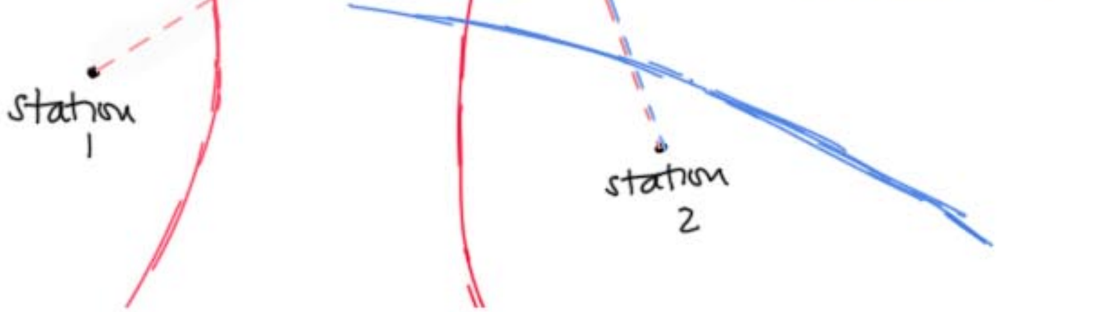
Definition: A <u>hyperbola</u> is the set of points for which the difference of the distances to two focus points is constant.



Applications of the hyperbola:

Trilateration problems, as with LORAN navigation:

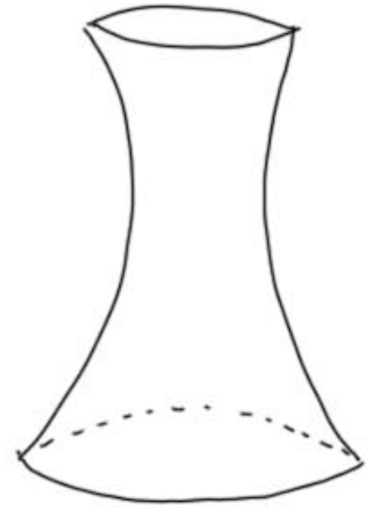
intersection of hyperbolas



and of the hyperboloid of one sheet (3D):

nuclear cooling towers:

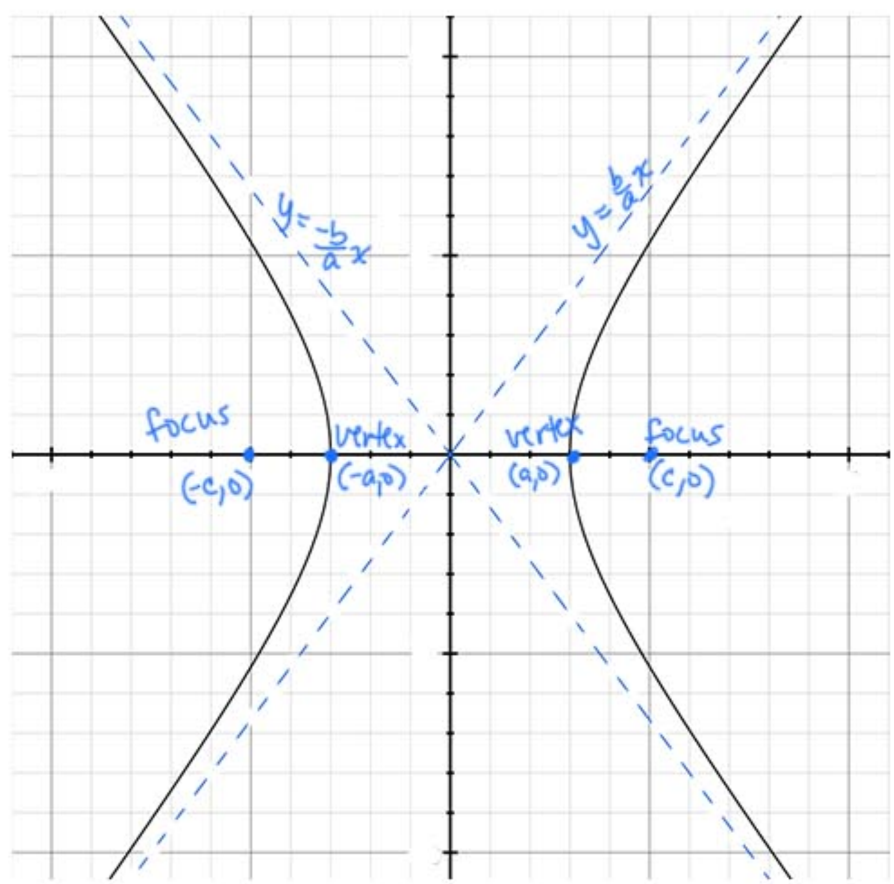
maximizes the strength of the structure while minimizing material used.



Centered at the origin:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

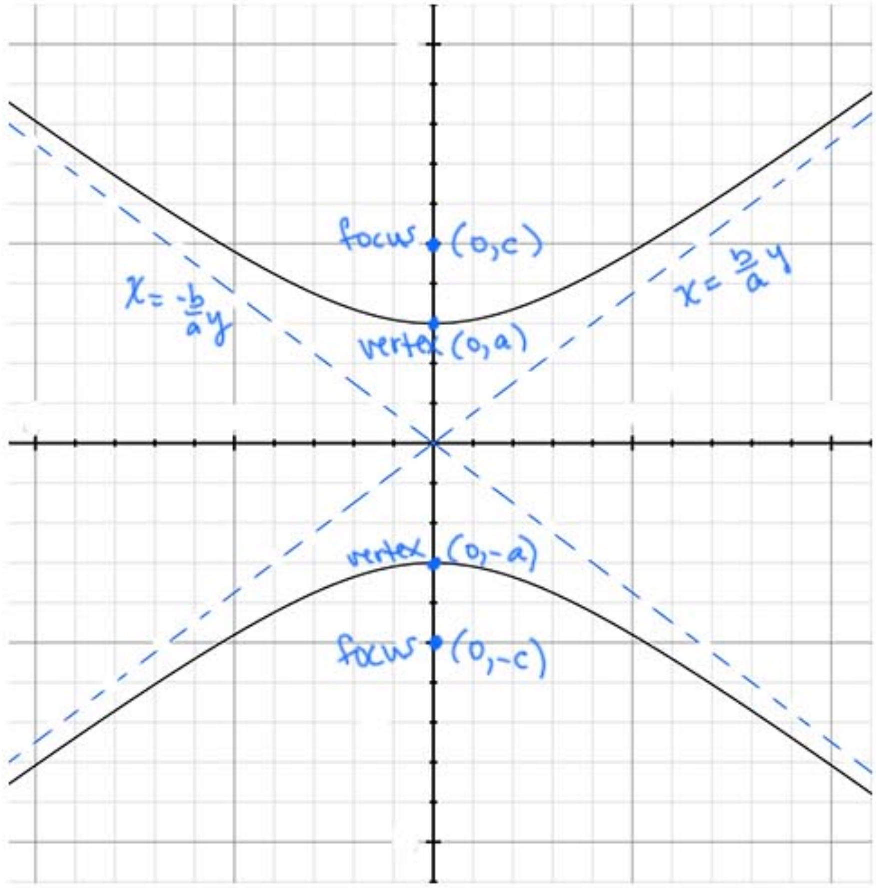
$$a^2 + b^2 = c^2$$



Similarly,

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$a^2 + b^2 = c^2$$



Ex. Find the foci, vertices, and asymptotes of the hyperbola: $4x^2 - y^2 = 16$

Divide by 16 : $\frac{4x^2}{16} - \frac{y^2}{16} = 1$

$$\frac{x^2}{4} - \frac{y^2}{16} = 1$$

$\therefore\ a = 2,\ b = 4$ $\quad\therefore c^2 = 2^2 + 4^2$

$= 20 \Rightarrow c = \sqrt{20} = 2\sqrt{5}$

Since $\frac{x^2}{4} - \frac{y^2}{16} = 1$

the positive term is x^2 $\Rightarrow$ vertices and foci along x axis

$\Rightarrow$ foci $(\pm 2\sqrt{5}, 0)$ vertices $(\pm 2, 0)$

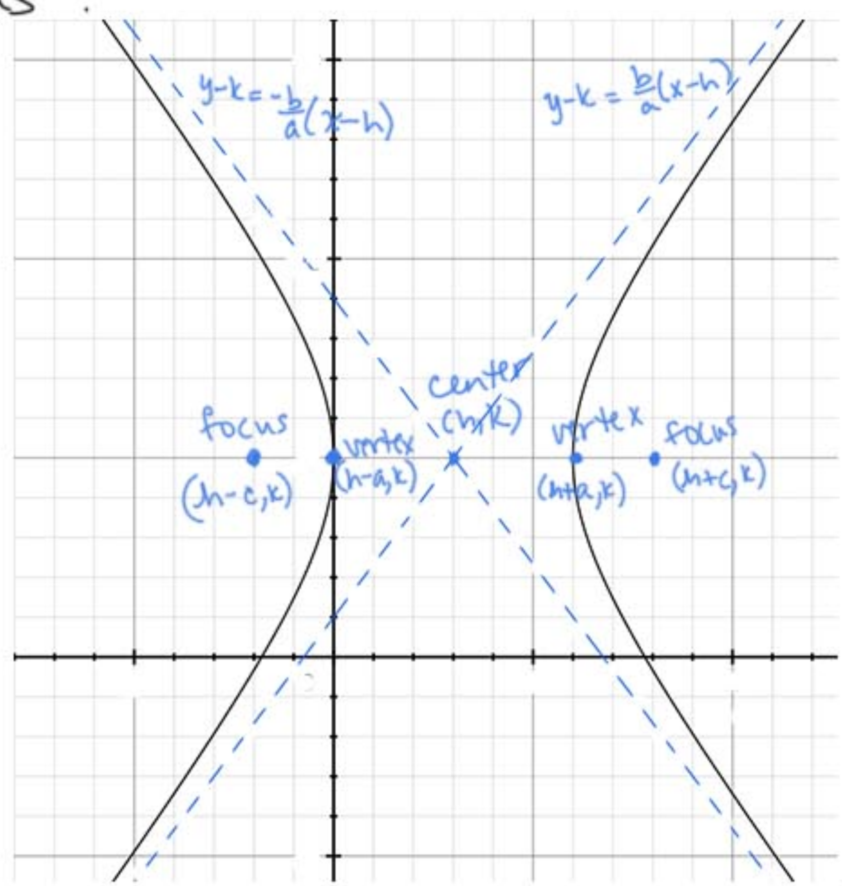
asymptotes $y = \pm \frac{4}{2} x$

$y = \pm 2x$.

Shifted hyperbolas :

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

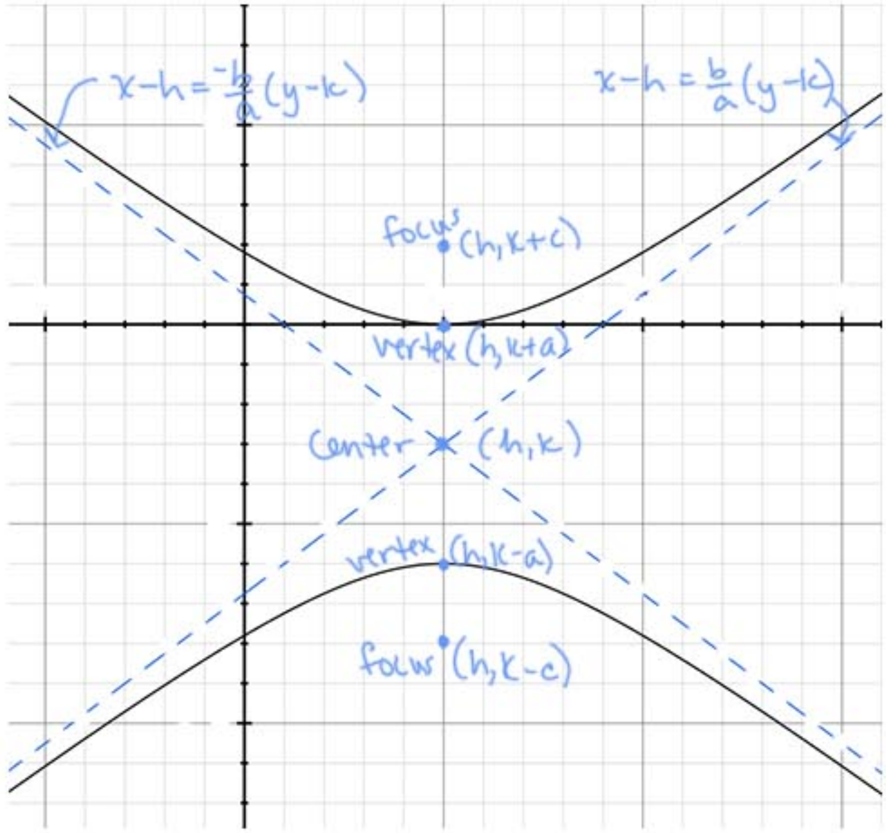
$a^2 + b^2 = c^2$



OR

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

$$a^2 + b^2 = c^2$$

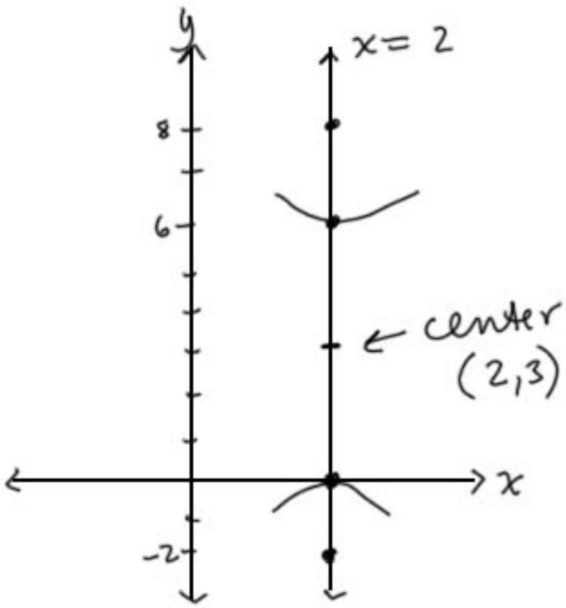


Ex. Find an equation for the hyperbola with foci $(2,-2)$ and $(2,8)$ vertices $(2,0)$ and $(2,6)$.

Work on this problem on your own

we see that the foci and vertices are

along $x=2$, so $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$



$\Rightarrow h=2,\ k=3$

$a=3,\ c=5$

$a^2+b^2=c^2$

$\Rightarrow b=4$

$$\frac{(y-3)^2}{9} - \frac{(x-2)^2}{16} = 1.$$

Notice: the form of these conic sections (when multiplied out) looks like:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

with $AC > 0 \Rightarrow$ ellipse

$AC < 0 \Rightarrow$ hyperbola

$AC = 0 \Rightarrow$ parabola