Math 20200
Calculus II

Lesson 21
Work

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Work

Work = Force · distance   when force is constant

if Force = $F(x)$ varies with position x, $a \le x \le b$

we can approximate $\text{Work} \approx \sum_{i=1}^{n} \underbrace{F(x_i)}_{\substack{\text{force}\\ \text{applied}\\ \text{at position } x_i}} \underbrace{\Delta x}_{\substack{\text{distance}\\ \text{moved}}}$

using $x_0 = a$
$x_i = a + i\Delta x$

$\Delta x = \frac{b-a}{n}$

$$\text{Work} = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i)\,\Delta x = \int_a^b F(x)\,dx$$

= work to move a particle over the interval $[a,b]$.

Types of work problems:

- work to move a particle
- work to compress or stretch a spring
- work to empty a tank of liquid
- work to lift a cable or rope

Ex. A particle is moved along the x-axis by a force that measures $\sqrt{4+x}$ lbs at at a point x feet from the origin.

Find the work done by moving the particle from the origin to a distance of 5 ft.

$$F(x) = \sqrt{4+x} \qquad \text{Work} = \int_0^5 \underbrace{\sqrt{4+x}}_{\text{lbs}}\ \underbrace{dx}_{\text{ft}} \quad \text{ft-lbs}$$

$u = 4+x$ $\quad$ $x=0$, $u=4$

$du = dx$ $\quad$ $x=5$, $u=9$

$$= \int_4^9 u^{1/2}\,du = \frac{2}{3}u^{3/2}\Big]_4^9$$

$$= \frac{2}{3}9^{3/2} - \frac{2}{3}4^{3/2} = \frac{2}{3}\cdot 27 - \frac{2}{3}\cdot 8$$

$$= \frac{2}{3}(27-8) = \frac{2}{3}\cdot 19 =$$

$$= \frac{38}{3} \text{ ft-lbs.}$$

Springs:

Hooke's Law: The force required to maintain a spring in a given position is proportional to the distance the spring is compressed/stretched.

$$\underbrace{F(x)}_{\text{force to maintain spring position}} = \underset{\substack{\uparrow \\ \text{spring constant}}}{k} \cdot \underbrace{x}_{\text{distance the spring is compressed or stretched from natural position}}$$

Then the work required to compress/stretch a spring d units from natural length is

$$W = \int_0^d kx \, dx$$

If a spring is already compressed (stretched) c units from natural length, the work required to compress (stretch) it further to d units from

natural length is $W = \int_c^d kx \, dx$.

Ex. A force of 4lbs is required to maintain a spring compressed 0.25 in from its natural length of 1.25 in. Determine the work needed to compress the spring from the natural length to a length of 0.85 in.

$4 = k(0.25) \Rightarrow k = \frac{4}{0.25} = 16.$

$\therefore F(x) = kx = 16x.$

$0.85 = \_\_$ in from natural length?

natural length $= 1.25$

$1.25 - 0.85 = \underline{0.4} = d.$

$$W = \int_0^{0.4} 16x \, dx = 8x^2 \Big|_0^{0.4} = 8(.4)^2$$

$= 1.28$ in-lbs (not a standard unit)

$\approx .107$ ft-lbs.

(replace the in with $\frac{1}{12}$ ft).

Ex. If the work needed to stretch a spring 1 ft beyond its natural length is 12 ft-lbs, how much work is needed to stretch it 9 inches beyond its natural length?

$\frac{3}{4}$ ft

we are given: $\int_0^1 kx \, dx = 12$ ft-lbs.

asked to find: $\int_0^{3/4} kx \, dx$.

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$$12 = \int_0^1 kx\,dx = k\frac{x^2}{2}\Big|_0^1 = k\left(\frac{1^2}{2} - \frac{0^2}{2}\right) = \frac{k}{2}$$

$$\therefore \; 12 = \frac{k}{2} \Rightarrow k = 24.$$

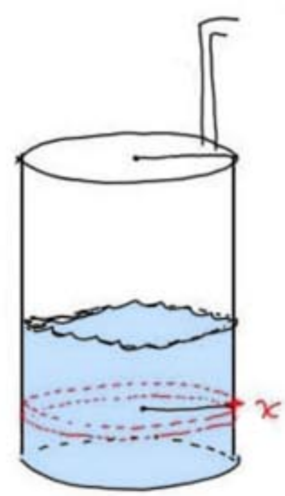
then $$\int_0^{3/4} kx\,dx = \int_0^{3/4} 24x\,dx = 12x^2\Big|_0^{3/4} = 12\left(\left(\tfrac{3}{4}\right)^2 - 0^2\right)$$

$$= 12 \cdot \frac{9}{16} = \frac{27}{4} \text{ ft-lbs.}$$

Tanks:

We are typically asked to find the work needed to pump the liquid to the top of the tank (or higher).

So we slice up the volume we are emptying, and:



force on the slice = mass · acceleration

= volume of slice · density of liquid · gravity

<u>either</u> $m^3 \cdot \dfrac{kg}{m^3} \cdot 9.81\ m/s^2 = N$

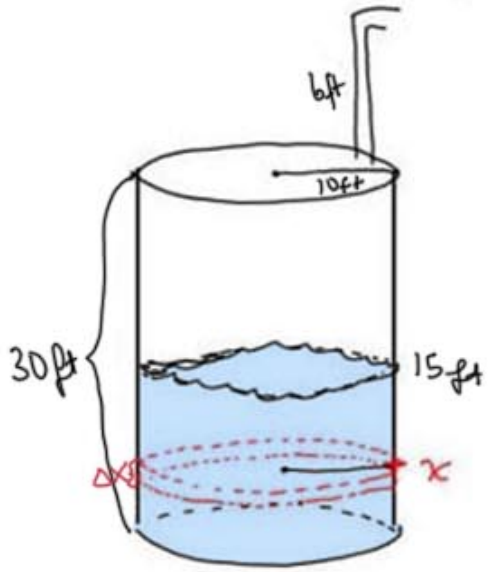
<u>OR</u> $ft^3 \cdot \dfrac{lb}{ft^3}$ ← already includes gravity $= lbs.$

Then

$$\text{work} = \int (\text{force on the slice})(\text{distance traveled by slice})$$

$$= \int (\text{volume of slice})(\text{density of liquid})(\text{gravity})(\text{distance traveled by slice})$$

Ex. A cylindrical water tank of radius 10 ft and height 30 ft is half-filled with water. How much work is required to pump all the water to 6 ft above the top of the tank? The density of water is 62.4 lb/ft^3.



let x = height from the bottom of the tank.

then $0 \le x \le 15$.

volume of slice = $\pi r^2 h$ (cylinder)

$= \pi (10)^2 \Delta x \rightarrow 100\pi\, dx$

density of water = $\underbrace{62.4}_{\text{3 significant figures}}$ lb/ft^3 (includes gravity)

distance traveled by the slice = top of spout $- x$

$= 36 - x$.

$$\therefore \quad W = \int_0^{15} 100\pi\, dx \cdot 62.4 \cdot (36-x)$$

$$= 6240\pi \int_0^{15} (36-x)\, dx$$

$$= 6240\pi \left[36x - \frac{x^2}{2}\right]_0^{15}$$

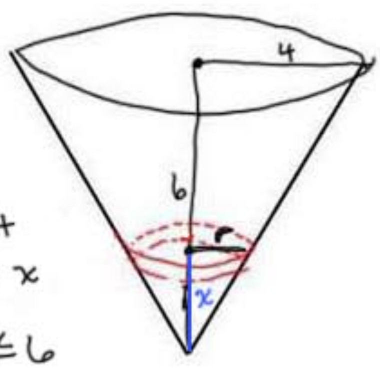
$$= 6240\pi \left[\left(540 - \frac{225}{2}\right) - (0-0)\right] = 2{,}670{,}000\pi \text{ ft-lbs}$$

rounded to 3 significiant figures

Ex. An open tank has the shape of a right circular cone. The tank is 8 m across the top and 6 m high. How much work is done in emptying the tank by pumping the water over the top edge? The density of water is 1,000 kg/m³.

slice at height x

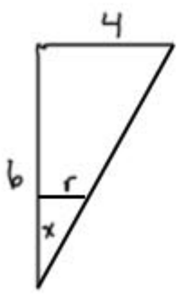
$0 \le x \le 6$



volume of slice $= \pi r^2 h$

$= \pi r^2 \Delta x$

we need r in terms of the height of the slice, x



Similar triangles:

$\frac{r}{x} = \frac{4}{6} \Rightarrow r = \frac{4}{6}x = \frac{2}{3}x$

$\therefore$ volume of slice $= \pi r^2 \Delta x$

$= \pi \left(\frac{2}{3}x\right)^2 \Delta x$

$= \frac{4}{9}\pi x^2 \Delta x \Rightarrow \frac{4}{9}\pi x^2 dx$

density of water: 1000 kg/m³ , gravity = 9.81 m/s² distance traveled by slice: $6-x$

$$\therefore \quad W = \int_0^6 \frac{4}{9}\pi x^2\,dx \cdot 1000 \cdot 9.81(6-x)$$

$$= \frac{4}{9}\pi\, 9810 \int_0^6 x^2(6-x)\,dx =$$

$$= \frac{4}{9}\pi\, 9810 \int_0^6 (6x^2 - x^3)\,dx$$

$$= \frac{4}{9}\pi\, 9810 \left[2x^3 - \frac{x^4}{4}\right]_0^6 =$$

$$= \frac{4}{9}\pi\, 9810 \left[\left(2\cdot 6^3 - \frac{6^4}{4}\right) - (0-0)\right] \approx 1{,}480{,}000 \text{ J}$$

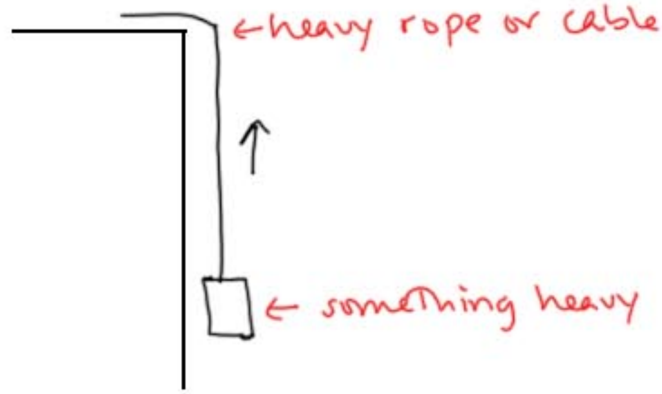
again, 3 significant figures from 9.81

$$kg \cdot \frac{m}{s^2} \cdot m$$

$$= N \cdot m = J.$$

Joule.

Cables and Ropes:



For cable/rope work problems, we have two methods:

method 1: slice the cable/rope and move each slice up to the top (similar to tank problems)

$$W = \int \begin{pmatrix} \text{force on slice} \\ \text{at height } x \end{pmatrix} (\text{distance traveled by slice})$$

method 2: at each height, find the force of the entire system, and move it Δx (dx) units (similar to moving particles at start of lesson)

$$W = \int (\text{force on entire system at height } x) \, dx$$

* rope, elevator, or non-leaky bucket can be solved either way

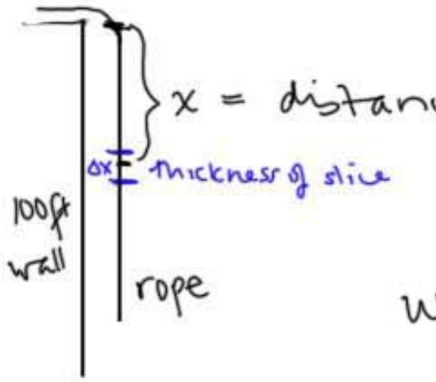
** leaky bucket must be solved by method 2.

Ex. An 80 ft rope that weighs 2 lb/ft hangs off the top of a 100 ft building.

a) How much work is done in pulling the rope to the top of the building?

b) How much work is done in pulling half the rope to the top of the building?

method 1: slice up the rope:



x = distance from top of bldg.

$$W = \int \begin{pmatrix}\text{force on slice} \\ \text{at height } x\end{pmatrix} (\text{distance traveled by slice})$$

a) force on the slice at height x $= 2\frac{\text{lb}}{\text{ft}} \cdot \Delta x \text{ ft} = 2\Delta x \text{ lbs}$

$\Rightarrow 2\,dx$

distance traveled by slice $= x$ ft

and the rope is 80 ft, so $0 \le x \le 80$

$$\therefore W = \int_0^{80} 2x\,dx = x^2 \Big|_0^{80} = 6400 \text{ ft-lbs}.$$

b) force on the slice at height x $= 2\frac{lb}{ft} \cdot \Delta x \text{ ft} = 2\Delta x$

$\Rightarrow 2\,dx$

distance traveled by the slice:

for $0 \le x \le 40$, same as above $= x$ ft.

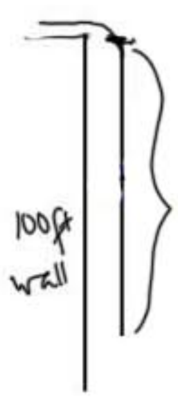
but for $40 \le x \le 80$, distance traveled is only 40 ft.

$$\therefore \quad W = \int_0^{40} 2x\,dx + \int_{40}^{80} 2(40)\,dx$$

$$= x^2\Big|_0^{40} + 80x\Big|_{40}^{80}$$

$$= 1600 + 80(80-40) = 1600 + 3200 = 4800 \text{ ft-lbs.}$$

OR method 2: $W = \int (\text{force on entire system at height } x)\,dx$



x = length of hanging rope

force on entire hanging rope $= 2\frac{lb}{ft} \cdot x \text{ ft} = 2x$ lbs

a) $0 \le x \le 80$ and $W = \int_0^{80} 2x\,dx = \ldots 6400$ ft-lbs.

b) $\underbrace{40 \le x \le 80}$ and $W = \int_{40}^{80} 2x\,dx = x^2 \Big|_{40}^{80} =$

only consider these x-values because the rope is only pulled halfway up.

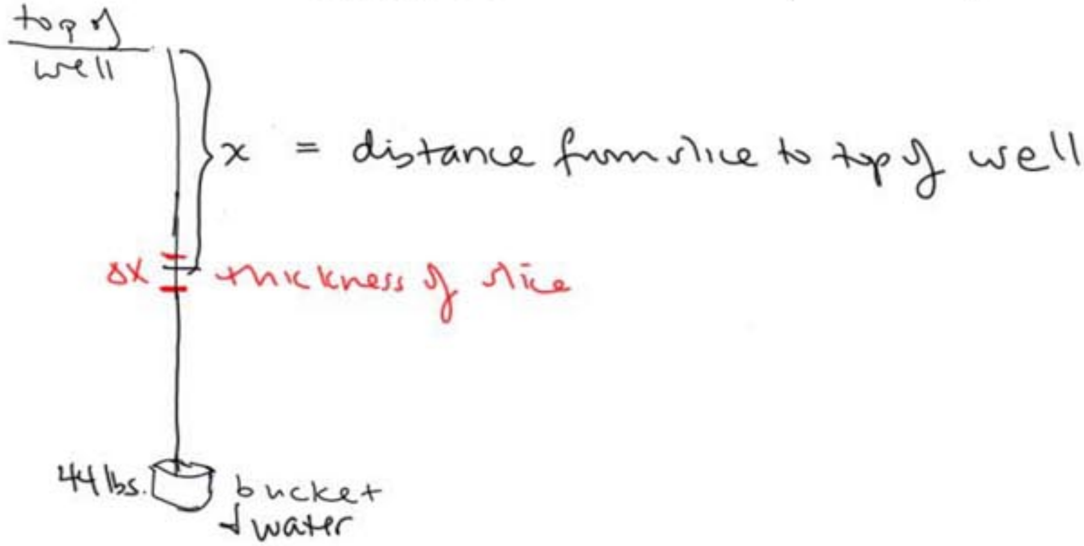
$= 6400 - 1600 = 4800$ ft-lbs

(same as above).

x starts at 80, and once we get to $x = 40$, we stop pulling.

Ex. A bucket that weighs 4 lbs and a rope that weighs 2 lb/ft are used to draw water from a well that is 80 ft deep. The bucket is filled with 40 lbs of water. Find the work done in pulling the bucket to the top of the well.

method 1: slice up the rope



$$W = \int \begin{pmatrix}\text{force on slice} \\ \text{at height } x\end{pmatrix} (\text{distance traveled by slice})$$

for $0 \le x \le 80$ (does not include the bucket of water)

$$\text{force} = \frac{2\text{lb}}{\text{ft}} \cdot \Delta x \text{ ft} = 2\Delta x \text{ lbs} \Rightarrow 2\,dx$$

$$\text{distance traveled} = x \text{ ft}$$

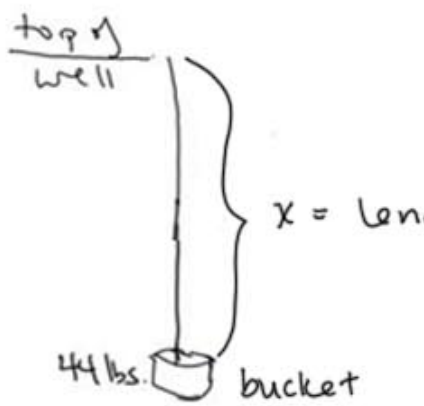
for $x = 80$ (the bucket of water)

force = 44 lbs, distance traveled is 80 ft.

$$\therefore \quad W = \int_0^{80} 2x\,dx + 44(80)$$

$$= x^2 \Big|_0^{80} + 44(80) = 6400 + 3520 = 9920 \text{ ft-lbs.}$$

OR Method 2: $W = \int (\text{force on entire system at height } x)\,dx$



x = length of rope = distance from bucket to top of well

$$0 \le x \le 80$$

force on the entire system at height x =

$$\underbrace{2\,\frac{\text{lb}}{\text{ft}} \cdot x \text{ ft}}_{\text{rope}} + \underbrace{44}_{\substack{\text{bucket}\\ \text{and water}}} \text{ lbs} = 2x + 44$$

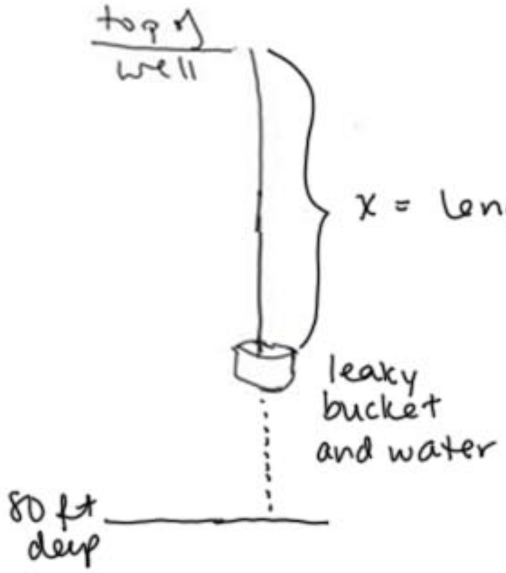
$$\therefore \quad W = \int_0^{80} (2x + 44)\, dx = x^2 + 44x \Big|_0^{80} =$$

$$= 6400 + 3520 = 9920 \text{ ft-lbs} \quad (\text{same answer}).$$

Ex. A bucket that weighs 4 lbs and a rope that weighs 2 lb/ft are used to draw water from a well that is 80 ft deep. The bucket is filled with 40 lbs of water and is pulled up at a rate of 2 ft/s, but water leaks out of a hole in the bucket at a rate of 0.2 lb/s. Find the work done in pulling the bucket to the top of the well.

** leaky bucket must use method 2:

$$W = \int (\text{force on entire system at height } x)\, dx$$



x = length of rope = distance from bucket to top of well

$$0 \le x \le 80$$

to find the force on the system at height x, we need to determine how much water is in the bucket at height x.

water is leaking at 0.2 lb/s
rope is pulled at 2 ft/s

so the bucket loses $.2$ lbs every 2 ft it is pulled

mathematically, $$\frac{0.2\,\frac{\text{lb}}{\text{s}}}{2\,\frac{\text{ft}}{\text{s}}} = \frac{0.2\text{ lb}}{2\text{ ft}} = 0.1\,\frac{\text{lb}}{\text{ft}}.$$

When the bucket is at height x (x feet from the top of the well), it has traveled $80-x$ ft and has lost $(.1)(80-x)$ lbs of water.

So the bucket contains $40-(.1)(80-x)$ lbs of water at height x

$$40-8+.1x = 32+.1x \text{ lbs}$$

$\therefore$ force on the system at height x =

force on rope + force on bucket + force on water

$$2\frac{\text{lb}}{\text{ft}}\cdot x \text{ ft} \quad + \quad 4 \text{ lbs} \quad + (32+.1x) \text{ lb.}$$

$$= 2x+4+32+.1x = 36+2.1x$$

and $$W = \int_0^{80}(36+2.1x)\,dx = 36x+\frac{2.1x^2}{2}\Big|_0^{80}$$

$$= 36(80)+\frac{2.1}{2}(80)^2 = 2880+6720 = 9600 \text{ ft-lbs.}$$