Math 20100
Calculus I

Lesson 8
The Derivative at a Point

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The Derivative at a Point.

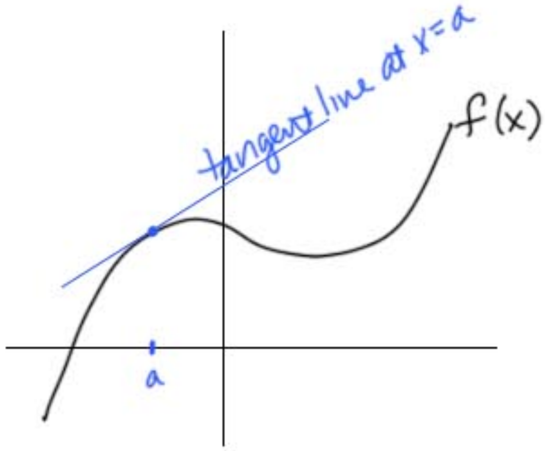
We are interested in finding the rate of change of a function $y = f(x)$ at a point, say $x = a$, ie. the change in f with respect to a change in x at $x = a$.

So it sounds like we are looking for the slope of $f(x)$.

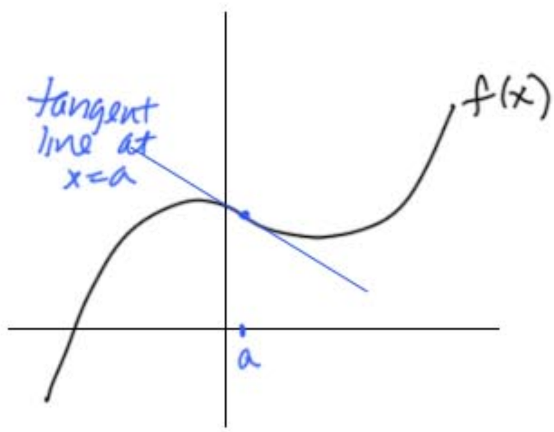
We say the <u>slope of $f(x)$ at $x = a$</u> is equal to the slope of the tangent line to $f(x)$ at $x = a$.

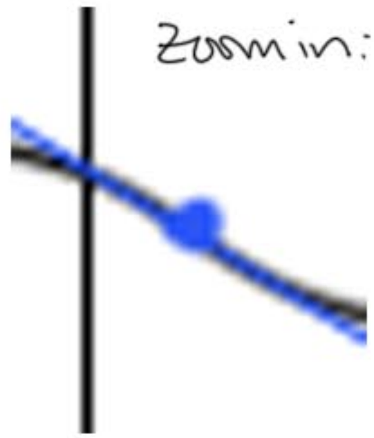
The tangent line to $f(x)$ at $x = a$ is the line touching the graph of $y = f(x)$ at $x = a$ with the same direction as $y = f(x)$ at $x = a$.

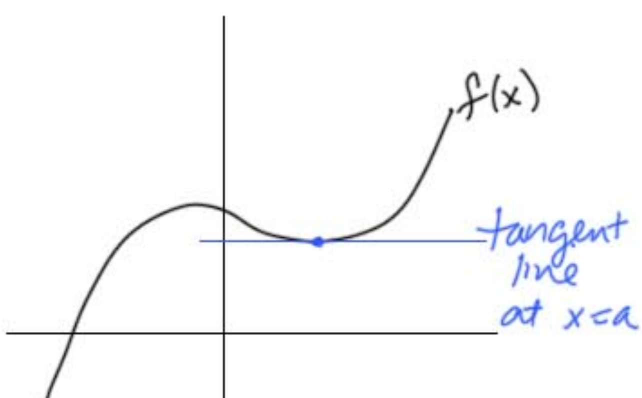
We can think of zooming in on the graph of $y = f(x)$ at $x = a$:



Zoom in:







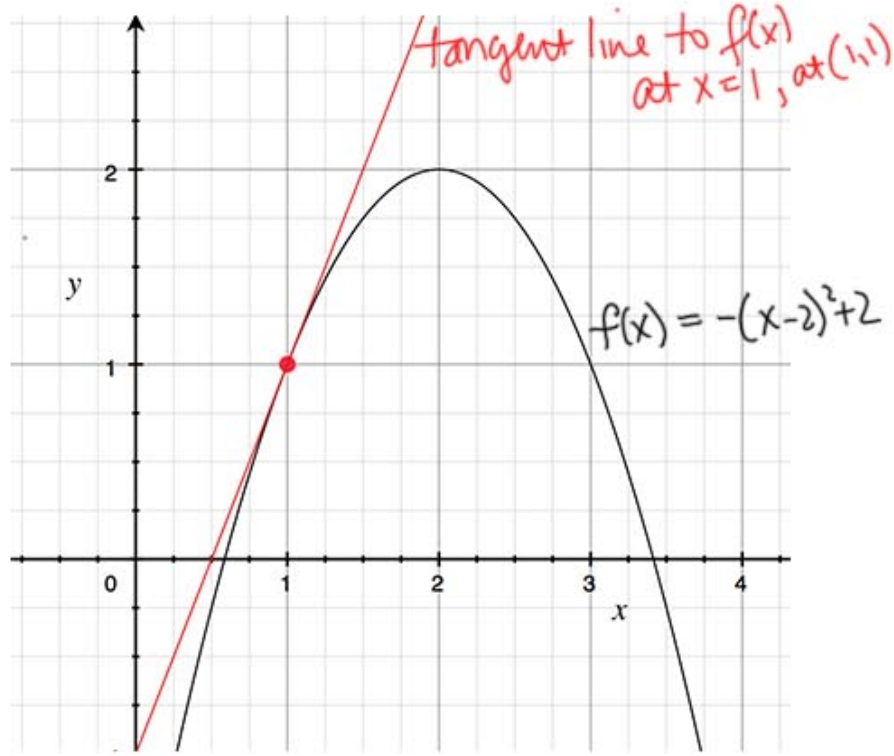
Zoom in

So to find the slope of the function $y=f(x)$ at $x=a$, we find the slope of the tangent line to f at $x=a$.

But we only know how to find the slope of a line for which we know two points. Here we only know one point, $(a, f(a))$ and we know the function $y=f(x)$. Let's see how we can use that to get the slope of the tangent line.

Consider the function $f(x) = -(x-2)^2 + 2$.

We want to find the slope of $f(x)$ at $x=1$, ie the slope of the tangent line to $f(x)$ at $x=1$.



Since we only know how to find the slope between two points, let's pick another point on $y=f(x)$, and take the slope of the secant line as an approximation to the slope of the tangent line:

through 2 points on a curve

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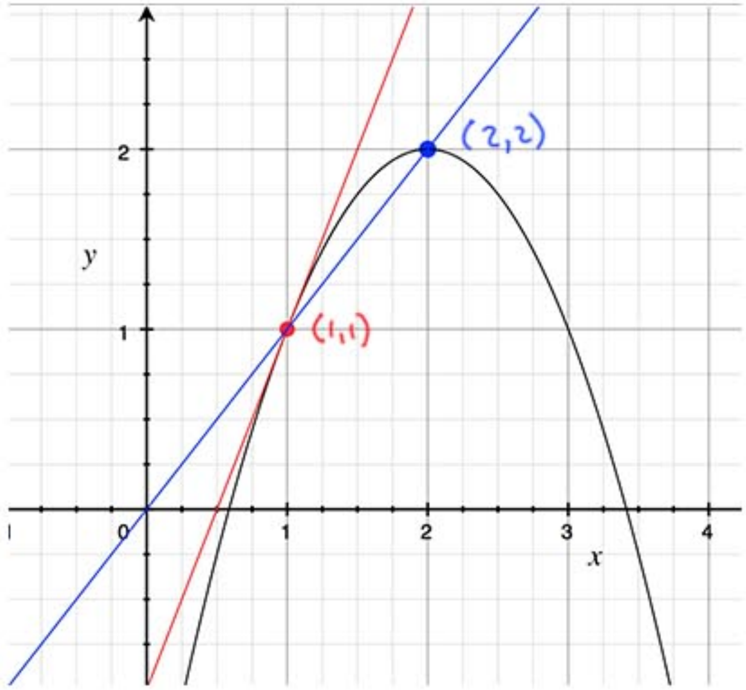
Slope of The secant line between (1,1) and (2,2)

is $m = \frac{2-1}{2-1} = 1$

This approximates The tangent slope. but we want a better approximation.



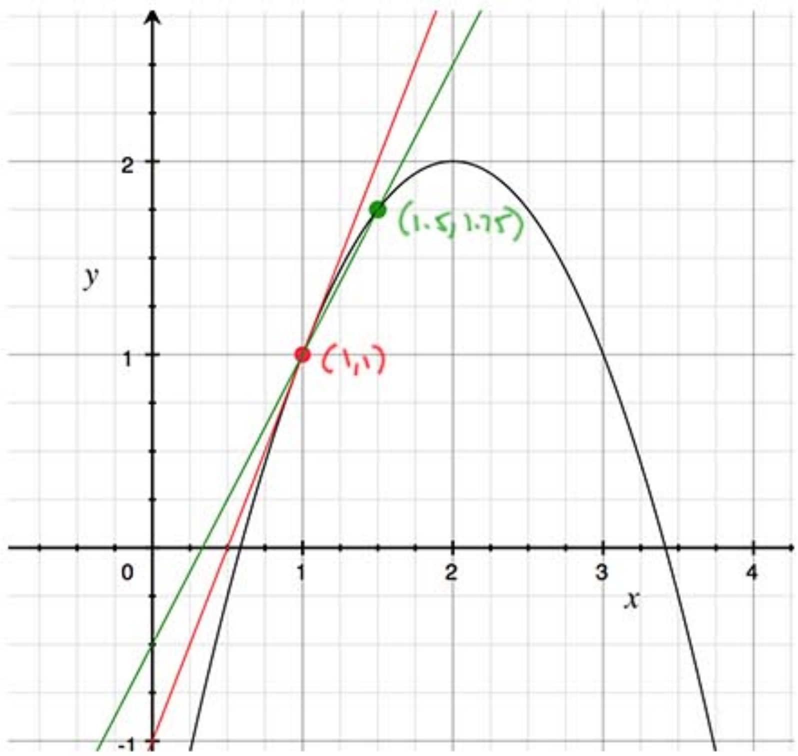
So we choose a point closer to $x=1$. say $x=1.5$

then $f(1.5) = -(1.5-2)^2 + 2 = -(-.5)^2 + 2 = -.25 + 2$
$= 1.75$

Slope of The secant line through (1,1) and (1.5, 1.75) is

$\frac{1.75-1}{1.5-1} = \frac{.75}{.5} = 1.5$

Still, we want a better approximation.



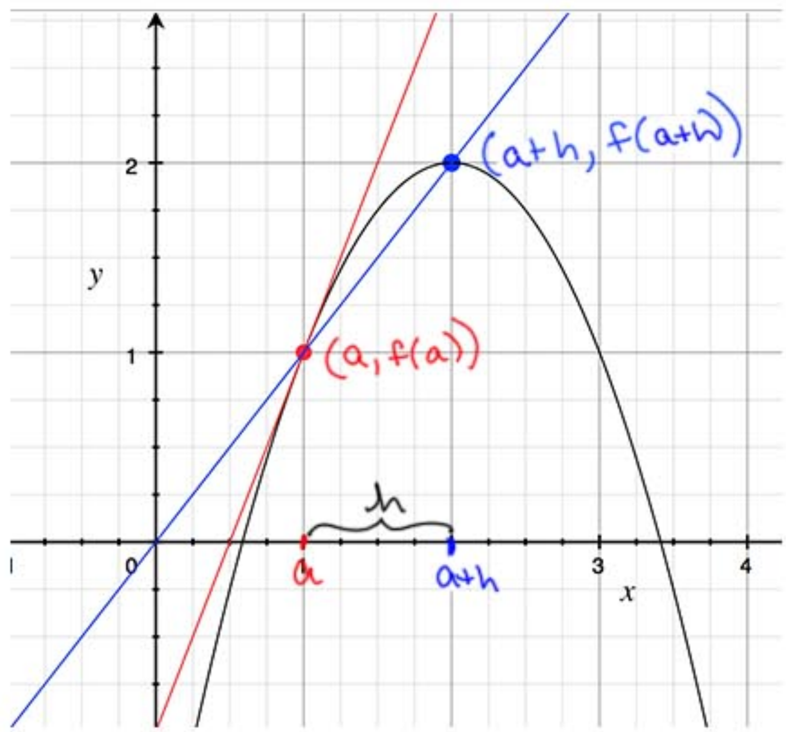
Notice, The slope of The tangent line = The limit of the slopes of The secant line as The x-value of the second point gets closer to 1.

Consider The general notation:

Slope of The secant line

$$= \frac{f(a+h)-f(a)}{a+h-a}$$

$$= \frac{f(a+h)-f(a)}{h}$$



$\therefore$ Slope of The tangent line to $f(x)$ at $x=a$ $= \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

and for our example, The slope of The tangent line to $f(x) = -(x-2)^2+2$ at $x=1$ is:

$= -x^2+4x-2$

$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{-(1+h)^2+4(1+h)-2-1}{h}$$ ← $f(1)=1$

$$= \lim_{h\to 0} \frac{-(1+2h+h^2)+4+4h-3}{h}$$

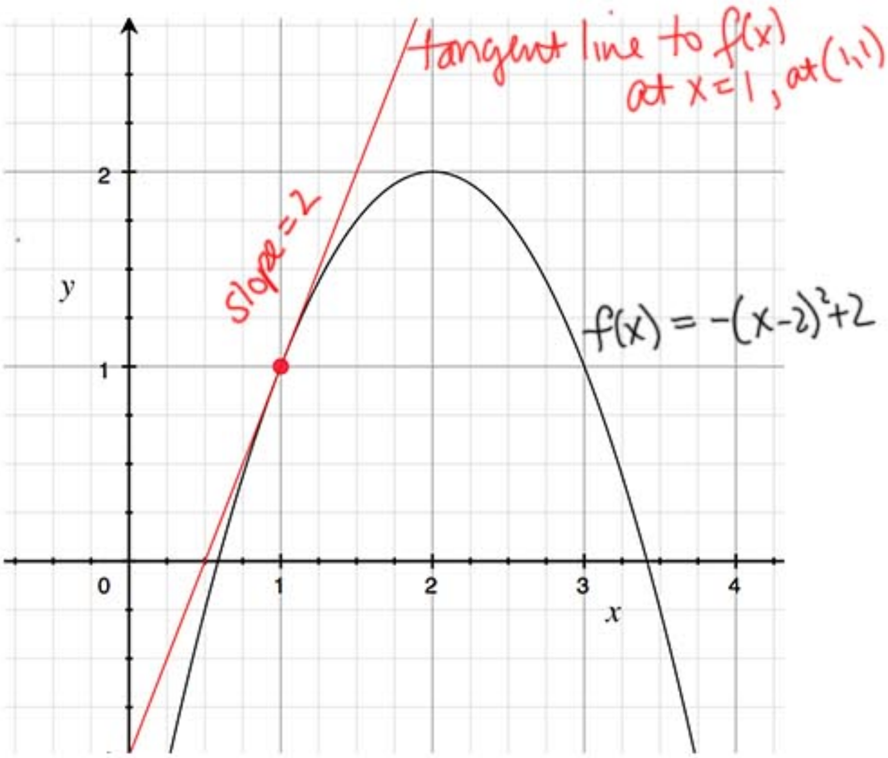
$$= \lim_{h\to 0} \frac{-\cancel{1}-2h-h^2+\cancel{4}+4h-\cancel{3}}{h}$$

$$= \lim_{h\to 0} \frac{2h-h^2}{h}$$

plug in $h=0$

$$= \lim_{h\to 0} \frac{\cancel{h}(2-h)}{\cancel{h}} = \lim_{h\to 0} 2-h = \boxed{2}$$

= slope of tangent line



$\therefore$ The slope of $f(x) = -(x-2)^2+2$ at $x=1$ is 2.

Also, what is an equation of that tangent line?

Work on this problem
on your own

The equation of a line is $y = mx+b$

or $y - y_1 = m(x - x_1)$

We have the point $(1,1) = (x_1, y_1)$ and the slope $m = 2$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x-1) \\ y - 1 &= 2x - 2 \\ +1 \quad & \quad\ +1 \\ y &= 2x - 1. \end{aligned}$$

or

$$\begin{aligned} y &= mx + b \\ 1 &= 2(1) + b \\ 1 &= 2 + b \\ -2 \quad & \ -2 \\ -1 &= b \end{aligned}$$

equation of the tangent line to $f(x) = -(x-2)^2+2$ at $x=1$. $\quad y = 2x - 1$.

* Note: we used secant slope approximations above to develop the idea of using the limit to get the slope of the tangent line. For future computations, we go straight to the limit computation.

* Also Note: we used the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{a+h-a} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

↑

slope of the secant line through $(a, f(a))$ and $(a+h, f(a+h))$, letting $h \to 0$.

We could also use:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \text{slope of the tangent line to } f(x) \text{ at } x = a$$

↑

slope of the secant line through $(a, f(a))$ and $(x, f(x))$

(more on this below)

We call these computations <u>the derivative of f at x = a</u> and we denote this by writing $f'(a)$ ("f prime of a").

$$\therefore \quad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

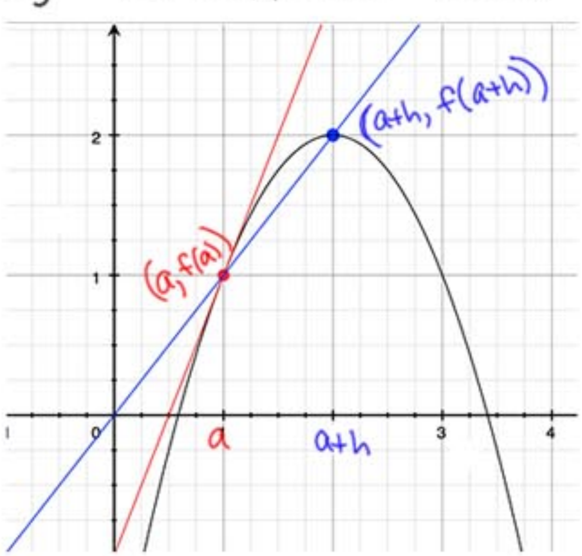
= the slope of $f(x)$ at $x = a$

= the slope of the tangent line to $f(x)$ at $x = a$

= the derivative of f at $x = a$

= the instantaneous rate of change of f at $x = a$

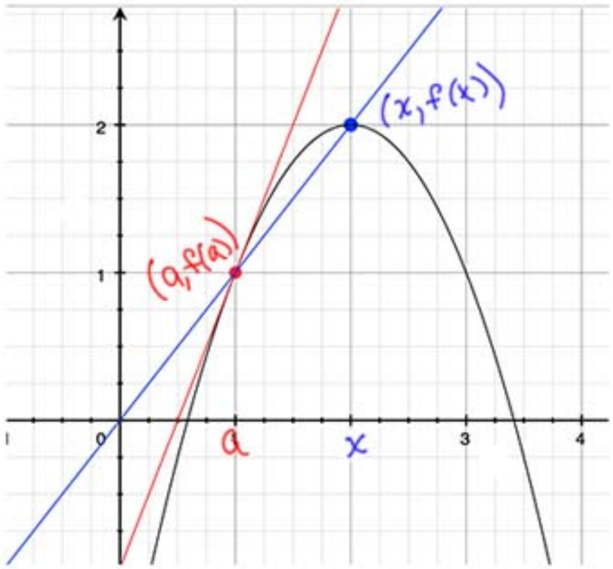
Comparing the different limit notations:



slope of the tangent line to $f(x)$ at $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

* usually more convenient



slope of the tangent line to $f(x)$ at $x=a$ is

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

* only convenient for $f(x)$ with high powers for which $f(a+h)$ would be messy:

Ex. For $f(x)=x^5$, find $f'(2)$.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^5 - 2^5}{h}$$

messy to multiply out

So we use:

$$f'(2) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{x^5-2^5}{x-2}$$

x^5-32

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(x^4+2x^3+4x^2+8x+16)}{\cancel{x-2}}$$

$$
\begin{array}{r}
1x^4+2x^3+4x^2+8x+16 \\
x-2\,\overline{)\,x^5+0x^4+0x^3+0x^2+0x-32} \\
-(x^5-2x^4) \\
\hline
2x^4 \\
-(2x^4-4x^3) \\
\hline
4x^3 \\
-(4x^3-8x^2) \\
\hline
8x^2 \\
-(8x^2-16x) \\
\hline
16x-32 \\
-(16x-32) \\
\hline
0
\end{array}
$$

$$= \lim_{x \to 2} x^4+2x^3+4x^2+8x+16$$

$$= 2^4+2\cdot 2^3+4\cdot 2^2+8\cdot 2+16$$

$$= 80 = f'(2).$$

Ex. Find the slope of $f(x) = \sqrt{x+1}$ at $x = 3$

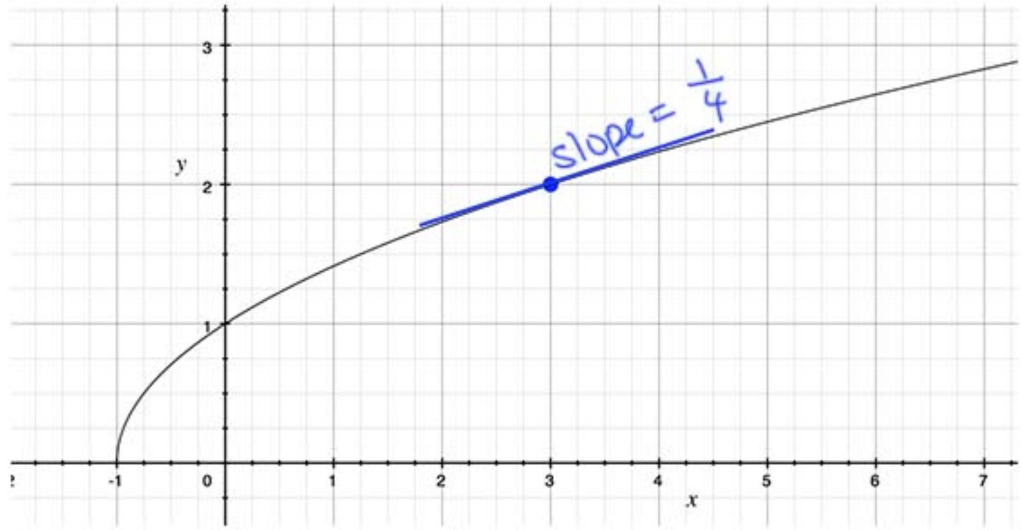
slope of $f(x)$ at $x=3$ = slope of tangent line to $f(x)$ at $x=3$ = $f'(3)$ =

$$= \lim_{h \to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h \to 0} \frac{\sqrt{3+h+1}-\sqrt{3+1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \lim_{h \to 0} \frac{\cancel{4}+h-\cancel{4}}{h(\sqrt{4+h}+2)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{4+h}+2)} = \lim_{h \to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}.$$

after cancelling the h, plug h=0



Ex. Find an equation of the tangent line to

$$g(x) = \frac{2x+1}{x+3} \quad \text{at} \quad x = 2.$$

We need the slope of the tangent line and a point on that line.

We are given $x = 2$ for the point of tangency,

so the point is $(2, g(2)) = (2, 1)$.

$$g(2) = \frac{2(2)+1}{2+3} = \frac{5}{5} = 1$$

slope at $x=2$ $= g'(2) = \lim_{h \to 0} \frac{g(2+h) - g(2)}{h}$

$$= \lim_{h \to 0} \frac{\frac{2(2+h)+1}{(2+h)+3} - 1}{h} = \lim_{h \to 0} \frac{\frac{4+2h+1}{5+h} - 1}{h}$$

$$= \lim_{h \to 0} \frac{\frac{5+2h}{5+h} - \frac{5+h}{5+h}}{\frac{h}{1}} = \lim_{h \to 0} \frac{5+2h-(5+h)}{(5+h)h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{(5+h)\cancel{h}} = \lim_{h \to 0} \frac{1}{5+h} = \frac{1}{5+0} = \frac{1}{5}$$

$\therefore$ $\frac{1}{5}$ = slope of tangent line to $g(x)$ at $x=2$.
at $(2,1)$.

$y - y_1 = m(x - x_1)$

$y - 1 = \frac{1}{5}(x-2)$

$y = \frac{1}{5}x - \frac{2}{5} + \frac{5}{5}$

$$\boxed{y = \frac{1}{5}x + \frac{3}{5}}$$

Ex. Find the slope of $f(t) = \frac{3}{\sqrt{t-1}}$ at $t=5$.

$$f'(5) = \lim_{h\to 0} \frac{f(5+h)-f(5)}{h} = \lim_{h\to 0} \frac{\frac{3}{\sqrt{5+h-1}} - \frac{3}{\sqrt{5-1}}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3}{\sqrt{4+h}} - \frac{3}{2}}{h} = \lim_{h\to 0} \frac{\frac{3}{\sqrt{4+h}}\cdot\frac{2}{2} - \frac{3}{2}\cdot\frac{\sqrt{4+h}}{\sqrt{4+h}}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{6-3\sqrt{4+h}}{2\sqrt{4+h}}}{\frac{h}{1}} = \lim_{h\to 0} \frac{6-3\sqrt{4+h}}{2h\sqrt{4+h}} \cdot \frac{6+3\sqrt{4+h}}{6+3\sqrt{4+h}}$$

$$= \lim_{h\to 0} \frac{36-9(4+h)}{2h\sqrt{4+h}\left(6+3\sqrt{4+h}\right)} = \lim_{h\to 0} \frac{-9\cancel{h}}{2\cancel{h}\sqrt{4+h}\left(6+3\sqrt{4+h}\right)}$$

$$= \lim_{h\to 0} \frac{-9}{2\sqrt{4+h}\left(6+3\sqrt{4+h}\right)} \underset{\text{plug } h=0}{=} \frac{-9}{2\sqrt{4}\left(6+3\sqrt{4}\right)} = \frac{-9}{4(6+6)}$$

$$= \frac{-9}{48} = -\frac{3}{16}$$

Rates of Change

We started the lesson by saying we were looking for the rate of change of $y=f(x)$, and we have been finding the rate of change of $f(x)$ at $x=a$

$= $ slope of $f(x)$ at $x=a$

$= $ slope of the tangent line to $f(x)$ at $x=a$

$= f'(a)$

To emphasize that this is the rate of change of f at one point, we call it the <u>instantaneous rate of change</u>.

We could also talk about the rate of change of f over the x-interval $[a,b]$: this is just the slope of the secant line through $(a,f(a))$ and $(b,f(b))$

$$\text{average rate of change of } f \text{ over } [a,b] = \frac{f(b)-f(a)}{b-a} = \text{slope of secant line through } (a,f(a)) + (b,f(b)).$$

We call This the average rate of change to emphasize that This is happening over an interval.

Ex. The displacement (in meters) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$, where t is measured in seconds.

a) find The average velocity (rate of change) over the time interval $[3,4]$

b) find The instantaneous velocity (rate of change) at $t = 4$.

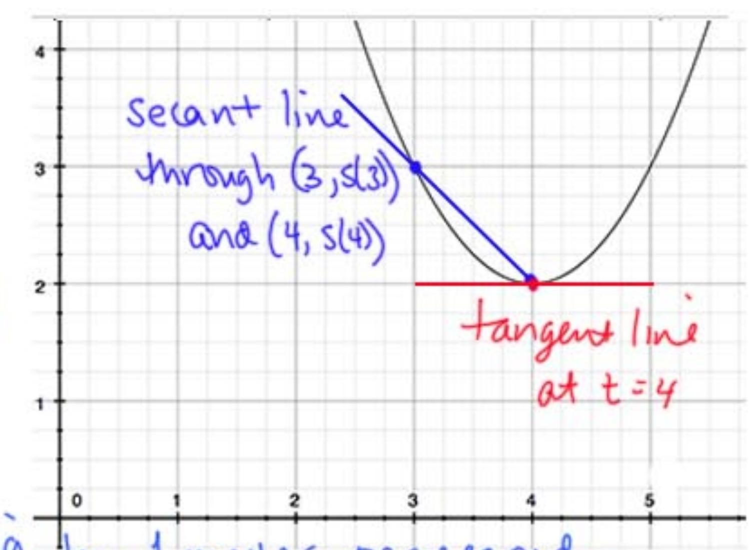
* Note: velocity $= \dfrac{\Delta \text{distance}}{\Delta \text{time}}$ = rate of change of distance

a) average velocity over $[3,4]$

$= \dfrac{s(4)-s(3)}{4-3} = \dfrac{2-3}{1} = \dfrac{-1}{1} = -1$

= slope of The secant line

$= -1$ m/s distance is decreasing by 1 meter per second.



b) instantaneous velocity at $t=4$ = $S'(4)$ =

$$\lim_{h\to 0} \frac{s(4+h)-s(4)}{h} = \lim_{h\to 0} \frac{(4+h)^2-8(4+h)+18-(4^2-8(4)+18)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{16}+\cancel{8h}+h^2-\cancel{32}-\cancel{8h}+\cancel{18}-\cancel{2}}{h} = \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h$$

$$= 0.$$

$= 0 \frac{m}{s}$ at $t=4$, the distance of the particle is not changing

(note, the particle is changing direction at $t=4$, so it stops to change direction).