Math 20100
Calculus I

Lesson 2
Important Functions to Know

Dr. A. Marchese, The City College of New York

Bookmarks have been added to this video at the following times:

Important Functions to Know

We learned in lesson 1 that functions are used to represent the relationship between quantities when one quantity depends on another.

In this lesson, we learn about some important functions. The simplest is the linear function, so we start with that.

Linear Functions

Linear functions are of the form

slope-intercept form

$$y = mx + b \qquad m = \text{slope}$$

$$f(x) = mx + b \qquad b = y\text{-intercept}$$

$$\text{slope } m = \frac{\Delta y}{\Delta x}$$ rate of change of y with respect to x.

So the slope of the line between points

(x_1, y_1) and (x_2, y_2) $\quad m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Also useful, the point-slope form of a line:

equation of the line through (x_1, y_1) with

slope m is $\quad y - y_1 = m(x - x_1)$.

Ex. The relationship between Fahrenheit (F)

and Celsius (C) temperatures is given by

a linear function. We know that the

freezing point of water is $0°C$ or $32°F$,

and the boiling point is $100°C$ or $212°F$.

a) Find the linear equation for F as a function

of C.

$x = C$ $\quad$ points $(0, 32)$

$y = F$ $\quad$ $(100, 212)$

$$m = \frac{212-32}{100-0} = \frac{180}{100} = \frac{9}{5}$$

(0, 32) is The y-intercept (F-intercept)

$b = 32$

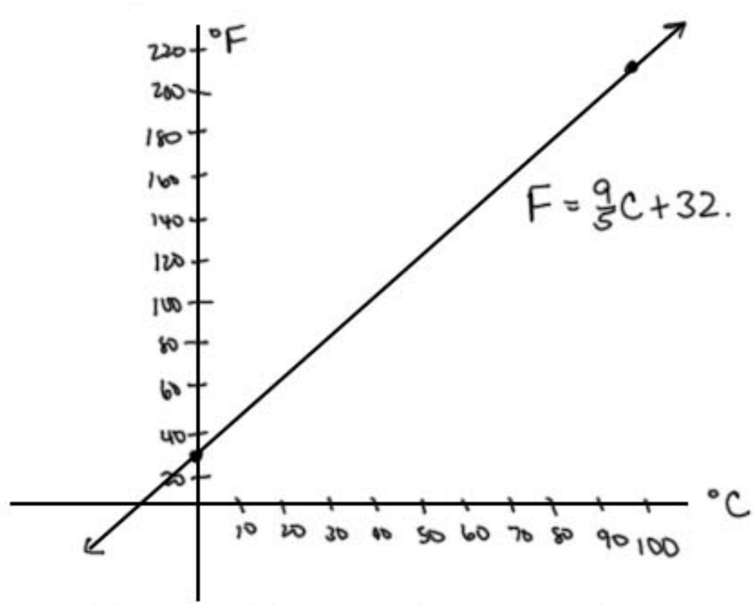
$$y = \frac{9}{5}x + 32 \quad \text{or} \quad F = \frac{9}{5}C + 32.$$

b) Find The Fahrenheit equivalent of 25°C.

$$F = \frac{9}{5}(25) + 32 = 45 + 32 = 77$$

$\therefore$ 77°F.

c) graph The function



Note: The slope $m = \frac{\Delta F}{\Delta C} = \frac{9}{5} \frac{\text{degrees F}}{\text{degrees C}}$

means that for every 5° increase in Celsius,

we get a 9° increase in Fahrenheit.

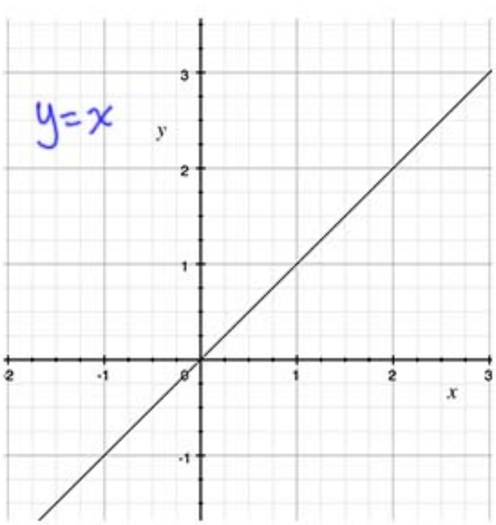
Power functions

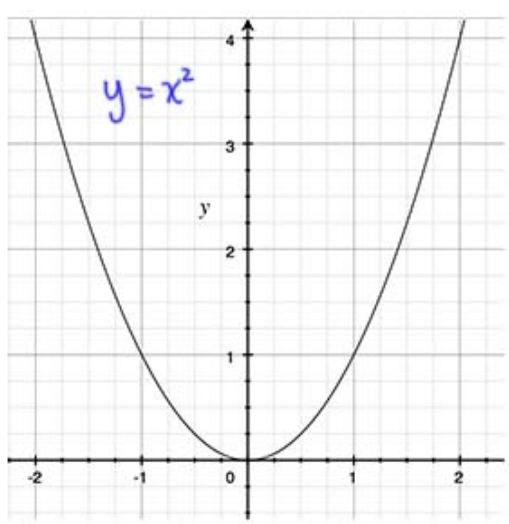
Power functions are of the form

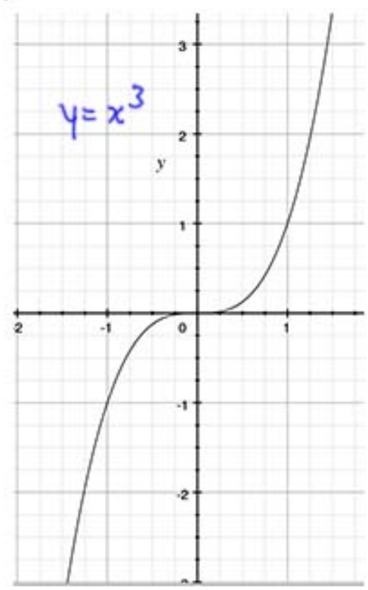
$$f(x) = x^a \quad (a \text{ is constant})$$

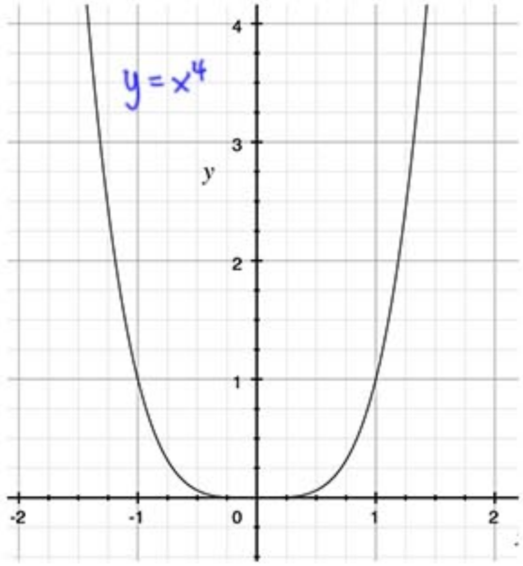
$$y = x^a$$

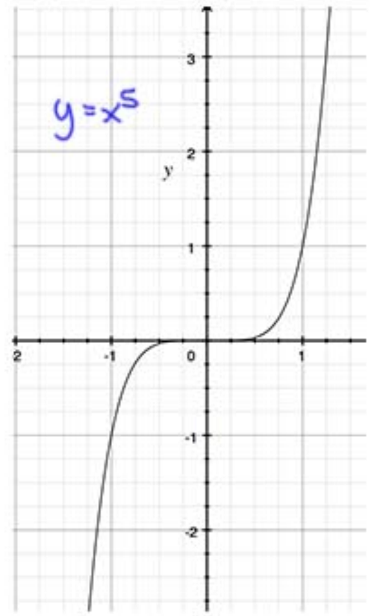
Ex. when $a = n$ positive integer

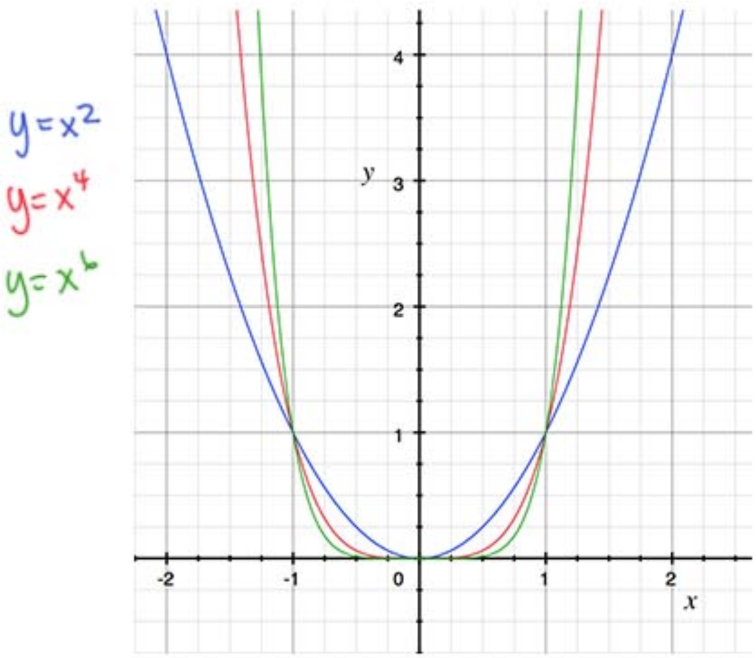


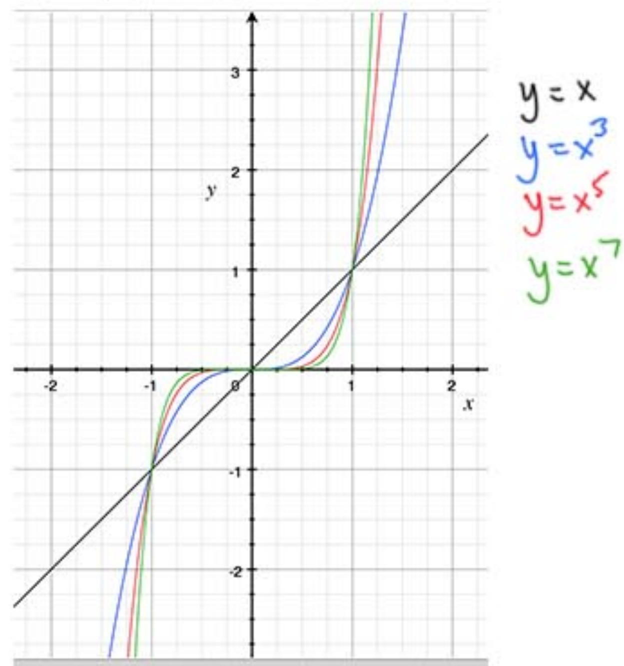






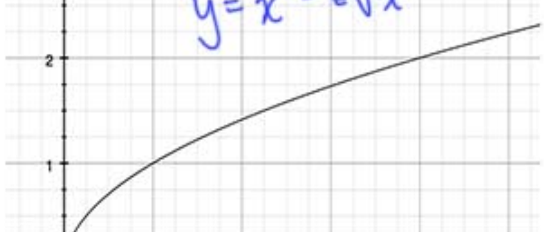


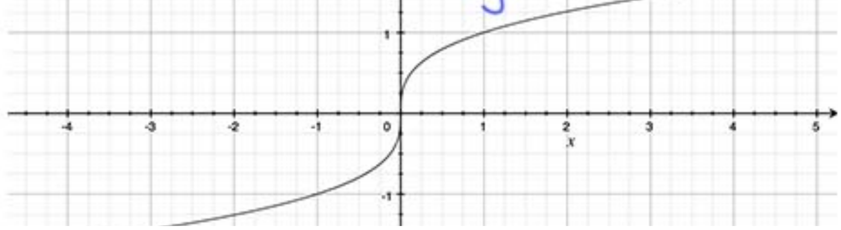


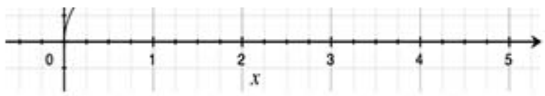


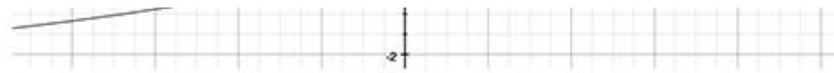
Ex. when $a = 1/n$ n is a positive integer

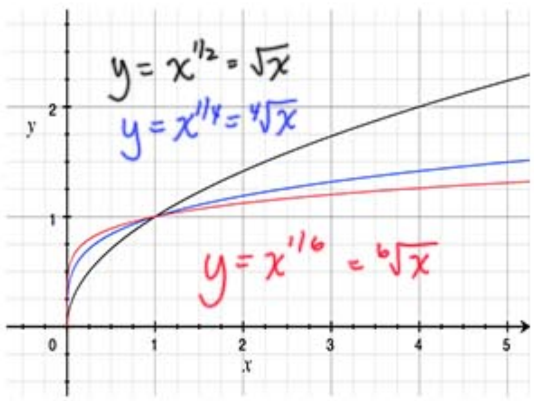
$y = x^{1/n} = \sqrt[n]{x}$ n^{th} root function

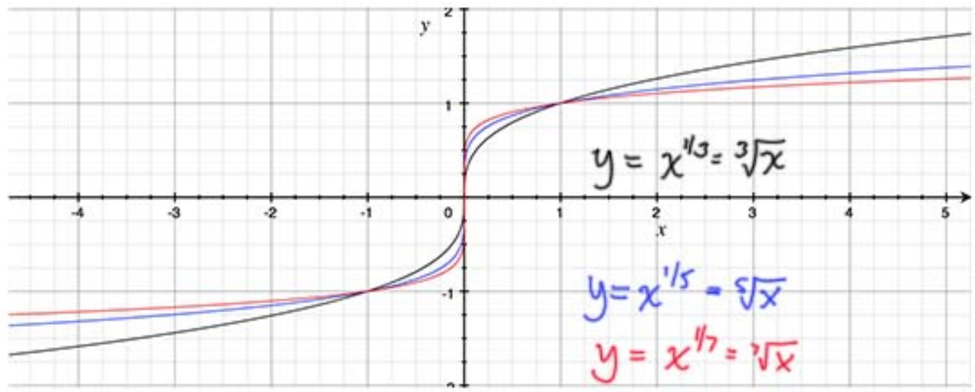




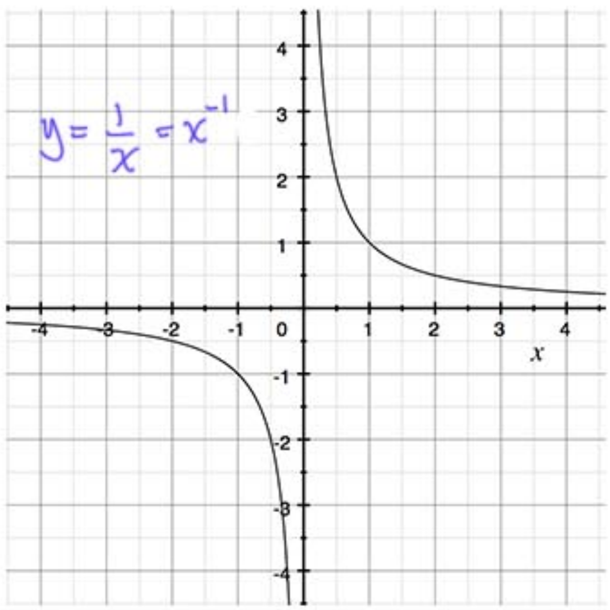


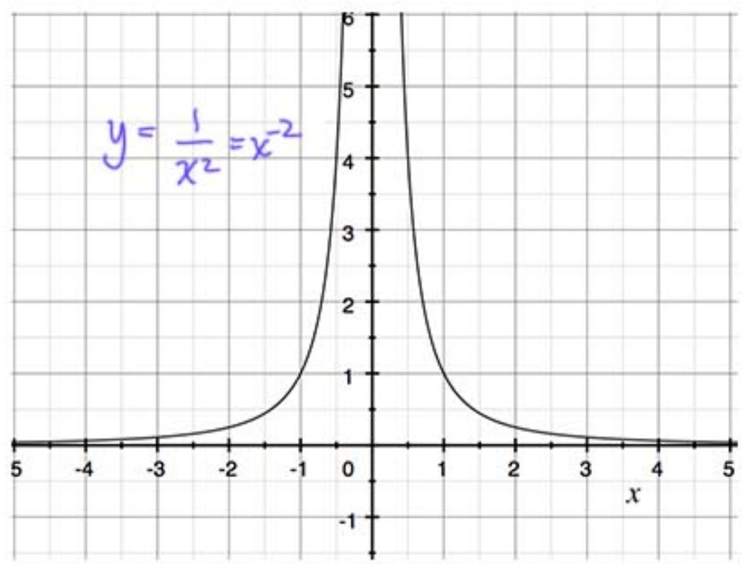






Ex. when $a = -1$ or $a = -2$



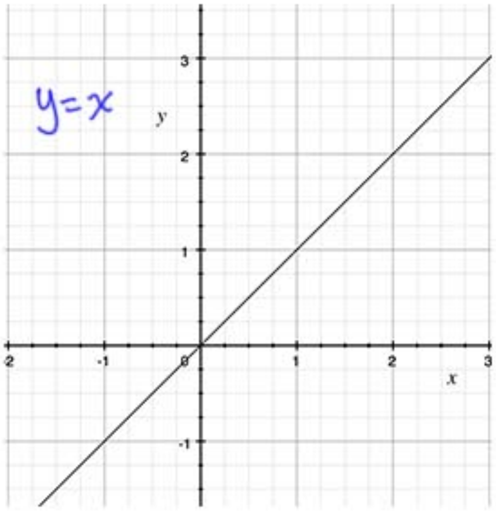


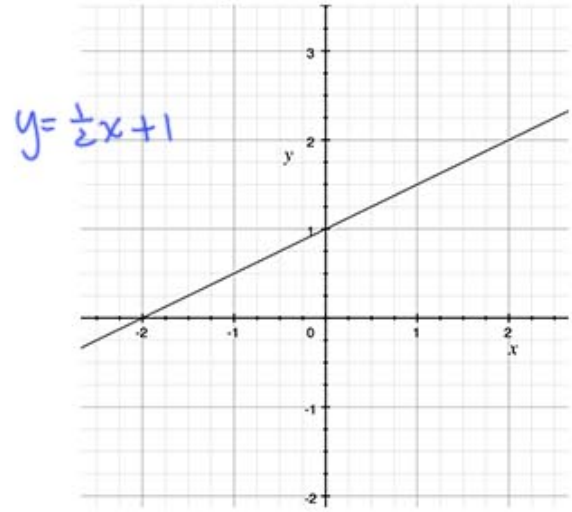
Polynomial Functions

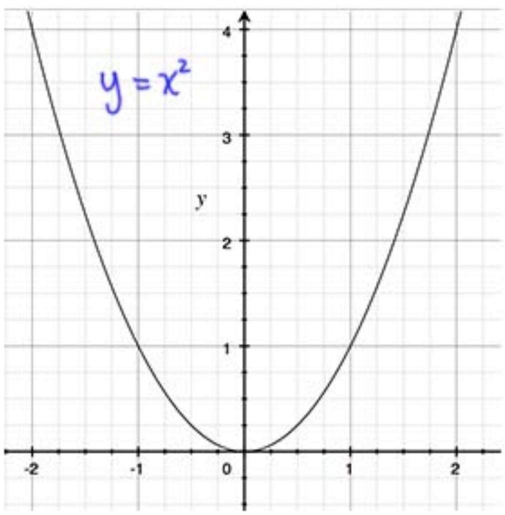
$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

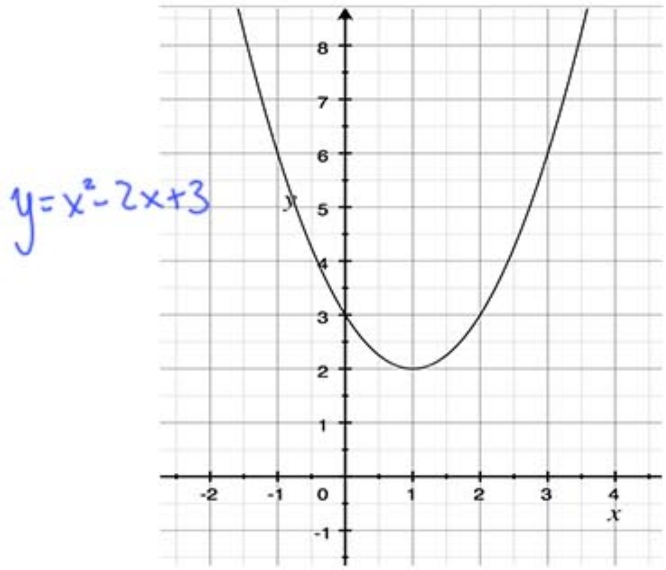
n is a positive integer

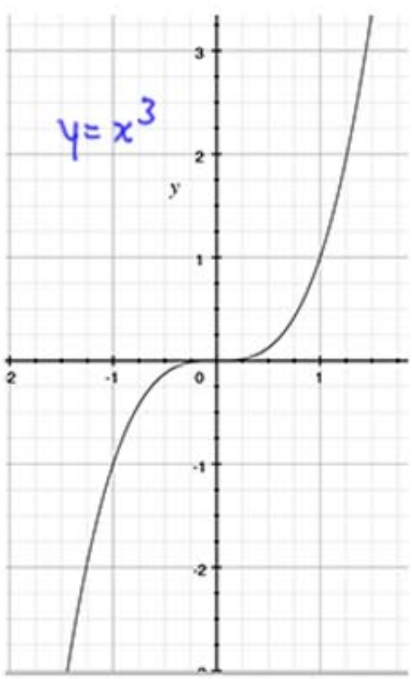
Power vs. Polynomial

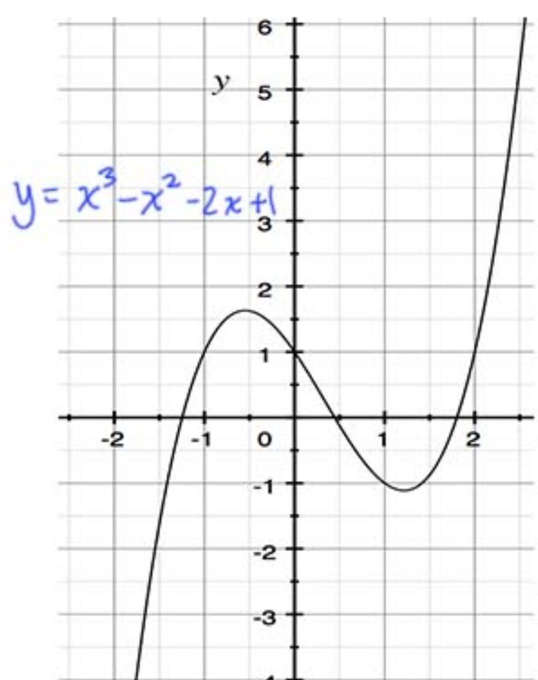


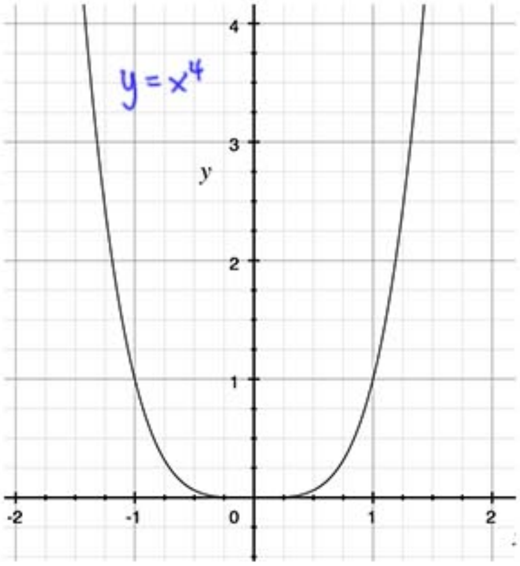


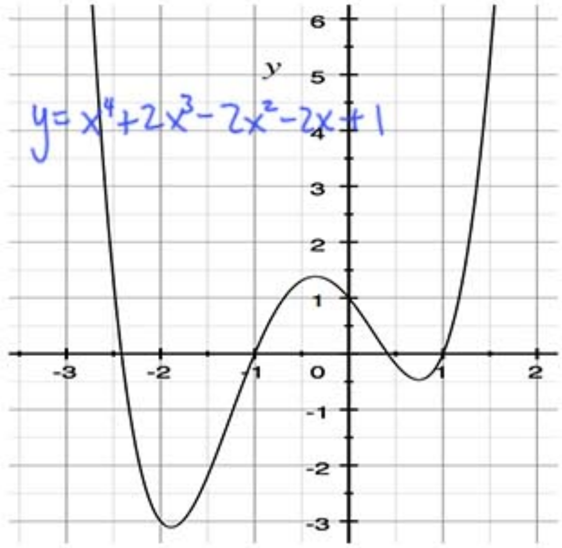










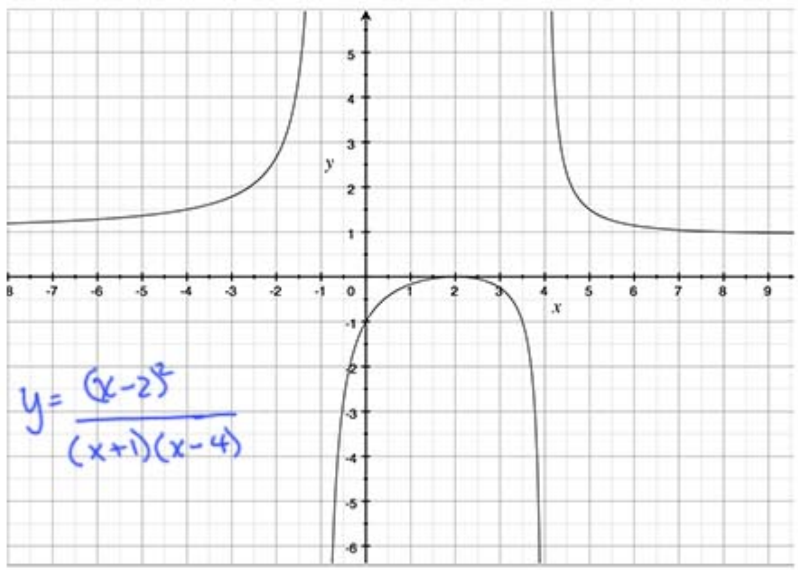


Rational Functions

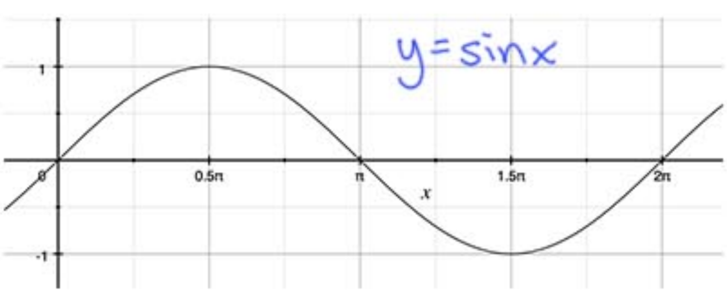
are of the form $\frac{P(x)}{Q(x)}$

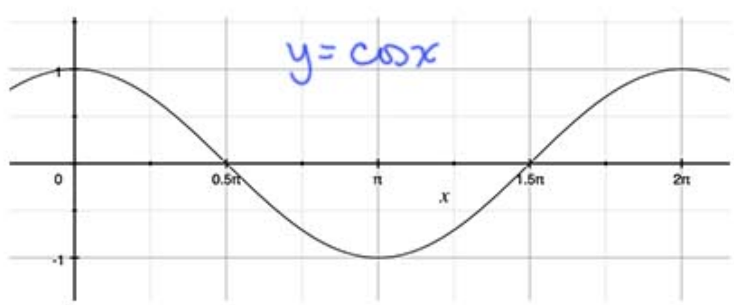
where $P(x)$ + $Q(x)$ are polynomials.

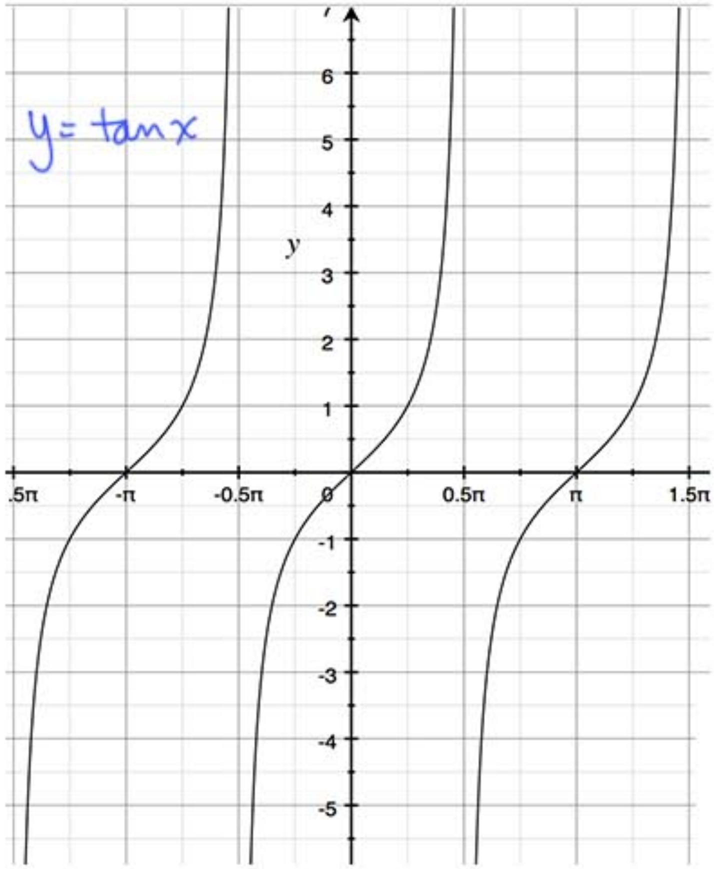
(more later in course with curve sketching)



Trigonometric Functions







Transformations of Functions

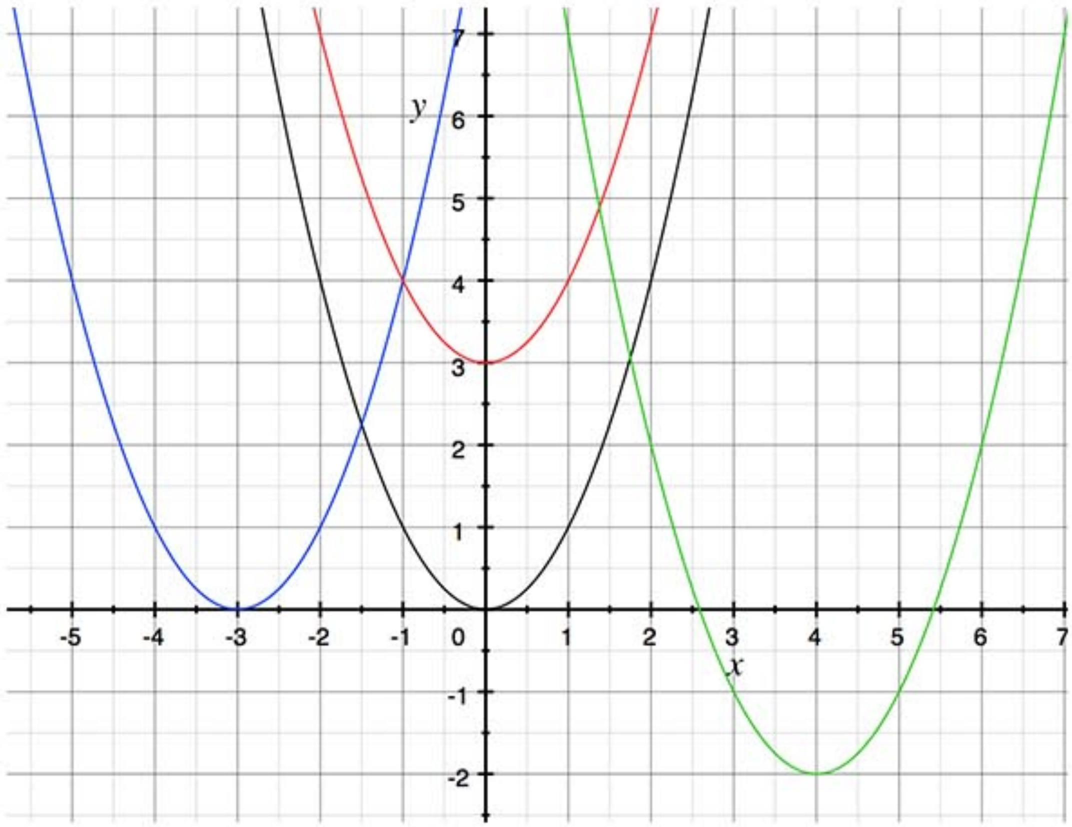
Vertical and Horizontal shifts: for $c > 0$

$y = f(x) + c$ shifts $y = f(x)$ up c units

$y = f(x) - c$ shifts $y = f(x)$ down c units

$y = f(x + c)$ shifts $y = f(x)$ left c units

$y = f(x - c)$ shifts $y = f(x)$ right c units



$y = x^2$

$y = x^2 + 3$

$y = (x+3)^2$

$y = (x-4)^2 - 2$

Stretching and Reflecting : for $c > 1$

$y = c f(x)$ stretches $y = f(x)$ vertically by a factor of c

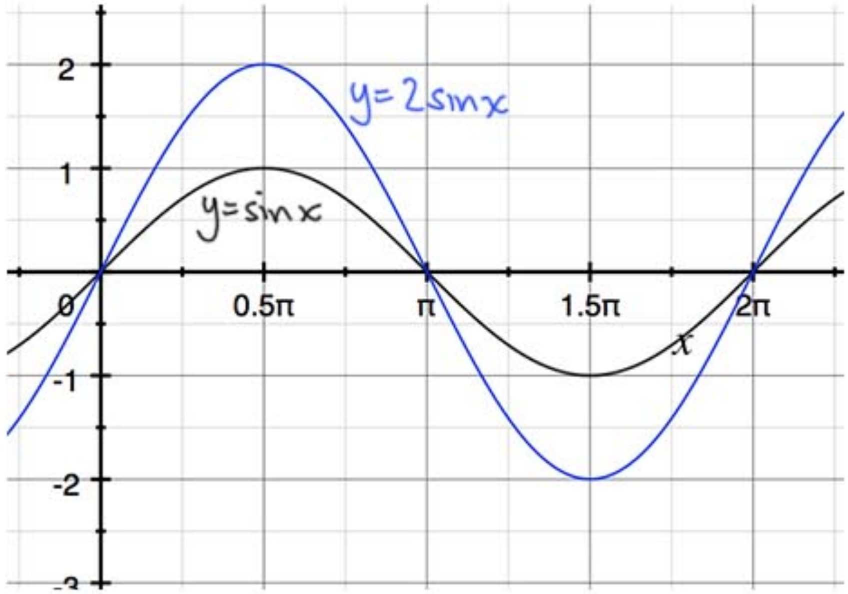
$y = \frac{1}{c} f(x)$ compresses $y = f(x)$ vertically by a factor of c

$y = f(cx)$ compresses $y = f(x)$ horizontally by a factor of c

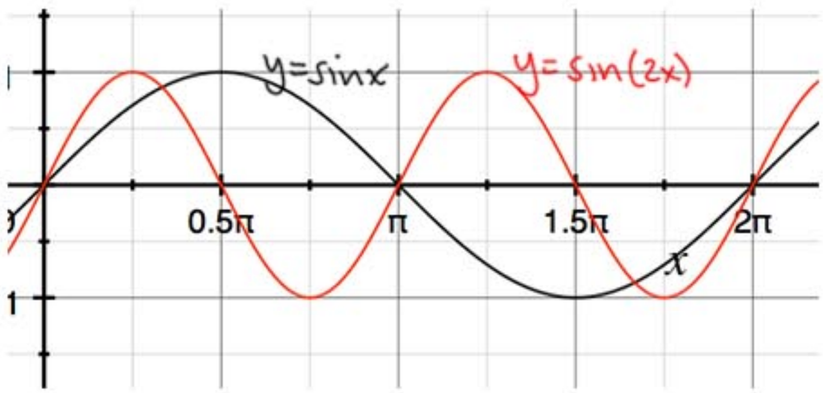
$y = f\left(\frac{x}{c}\right)$ stretches $y = f(x)$ horizontally by a factor of c

$y = -f(x)$ reflects $y = f(x)$ through the x-axis

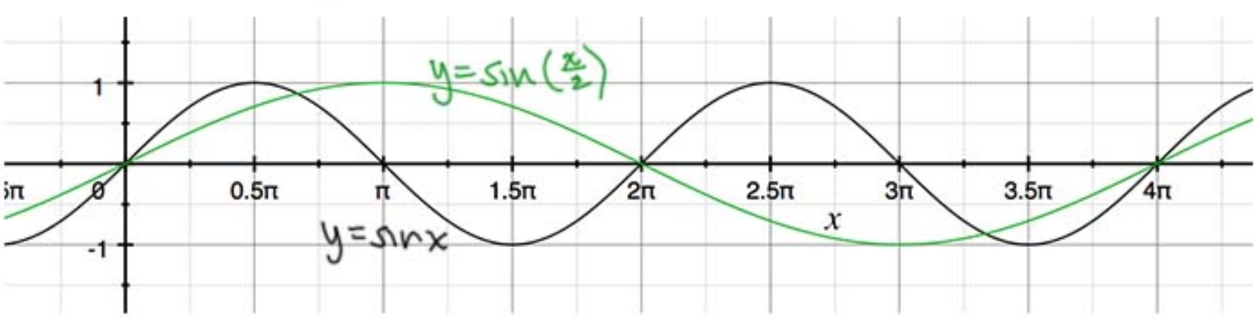
$y = f(-x)$ reflects $y = f(x)$ through the y-axis vertical stretch by a factor of 2



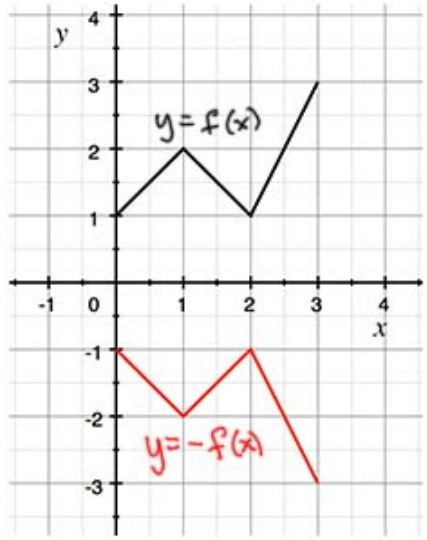
horizontal compression by a factor of 2



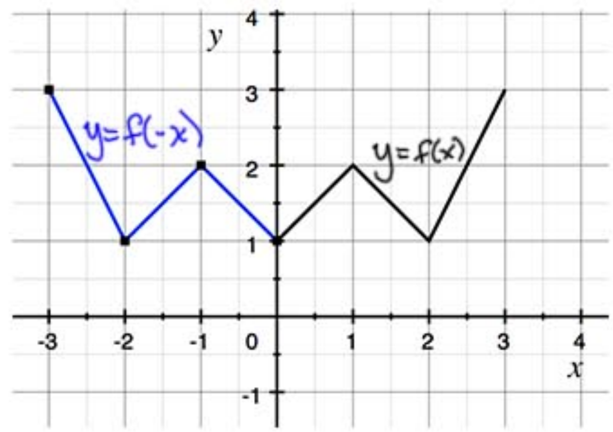
horizontal stretch by a factor of 2



reflection through the x-axis



reflection through the y-axis



Combinations of Functions

When combining functions by addition, subtraction, multiplication or division, it is important to recognize the proper domain.

For $y = f(x) + g(x)$
$y = f(x) - g(x)$
$y = f(x)g(x)$

the domain is the intersection (domain of f) $\cap$ (domain of g)

For $y = \dfrac{f(x)}{g(x)}$, we start with

(domain of f) $\cap$ (domain of g)

and remove any values for which

$g(x) = 0$.

Ex. $f(x) = \sqrt{x-2}$ $\quad g(x) = \dfrac{x}{x-3}$

then $h(x) = f(x) + g(x)$

$= \sqrt{x-2} + \dfrac{x}{x-3}$ has domain

$x - 2 \geq 0 \quad \cap \quad x \neq 3$

$x \geq 2 \quad \cap \quad x \neq 3$

$[2,3) \cup (3, \infty)$

and $r(x) = \dfrac{f(x)}{g(x)} = \dfrac{\sqrt{x-2}}{\frac{x}{x-3}} = \dfrac{(\sqrt{x-2})(x-3)}{x}$

starts with $[2,3) \cup (3,\infty)$ and removes (if necessary)

$x = 0$

$\Rightarrow [2,3) \cup (3, \infty)$.

* Note that $x = 3$ is <u>not</u> in the domain of $\frac{f(x)}{g(x)}$

Compositions of Functions

For functions f and g, The composition

$$(f \circ g)(x) = f(g(x)) \quad \text{"f of g of x"}$$

Ex. $f(x) = \sqrt{x} \qquad g(x) = x - 3$

$$f(g(x)) = f(x-3) = \sqrt{x-3}$$

$$g(f(x)) = g(\sqrt{x}) = \sqrt{x} - 3$$

$$g(g(x)) = g(x-3) = (x-3) - 3 = x - 6$$

* The domain of $f(g(x))$ must have:

x in The domain of g

$g(x)$ in The domain of f.

Ex above. $f(x) = \sqrt{x} \qquad g(x) = x - 3$

domain of f: $x \geq 0$ domain of g: $\mathbb{R}$

$f(g(x)) = \sqrt{x-3}$ need $x \in \mathbb{R}$

also need $\underbrace{x-3}_{g(x)} \geq 0$ (in domain of f)

$\therefore$ $x \geq 3$ domain of $f(g(x))$

Ex. $f(x) = x^2+2$ $g(x) = \sqrt{x}$

domain of f: $\mathbb{R}$ domain of g: $x \geq 0$

$f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 + 2 = \underbrace{x+2}$

looks like domain should be $\mathbb{R}$

BUT remember this is a composition, and x must first be pluggend into g. So the domain needs: $x \geq 0$ (from domain g)

also be sure $\underbrace{\sqrt{x}}_{g(x)}$ in domain of f

$\therefore$ domain of $f(g(x))$ is $x \geq 0$.

Ex. $f(x) = \frac{2}{x}$ $\quad g(x) = \frac{3x-6}{2x+1}$

domain of f: $x \neq 0$ $\qquad$ domain of g: $x \neq -\frac{1}{2}$

$$f(g(x)) = f\left(\frac{3x-6}{2x+1}\right) = \frac{2}{\frac{3x-6}{2x+1}} = \frac{2(2x+1)}{3x-6}$$

need $x \in$ domain of g $\Rightarrow$ $x \neq -\frac{1}{2}$

also need $g(x) \in$ domain of f $\Rightarrow$ $\frac{3x-6}{2x+1} \neq 0$

$\Rightarrow$ $x \neq 2$

$\therefore$ domain of $f(g(x))$ is $x \neq -\frac{1}{2}, x \neq 2$

or $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, 2) \cup (2, \infty)$.