Math 20100
Calculus I

Lesson 9
The Derivative Function

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The Derivative Function

In Lesson 8 we saw the derivative of f at a point $x = a$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

= the slope of $f(x)$ at $x = a$

= the slope of the tangent line to $f(x)$ at $x = a$

= the derivative of f at $x = a$

= the instantaneous rate of change of f at $x = a$

Now, instead of specifying a point $x = a$, we leave x in the computation (to represent any x-value):

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \text{the derivative function } f'(x).$$

$\therefore$ we have :

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ "definition of the derivative"

$$= \text{the slope function for } f(x)$$

$$= \text{the derivative of } f(x).$$

Ex. For $f(x) = x^3 - x^2 - x + 1$, find $f'(x)$. Use $f'(x)$ to find the slope of f at $x = -1$, $x = 0$, and $x = 2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - (x+h)^2 - (x+h) + 1 - (x^3 - x^2 - x + 1)}{h}$$

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$$(x+h)^2 = x^2+2xh+h^2$$

$$(x+h)^3 = (x+h)(x+h)^2 = (x+h)(x^2+2xh+h^2)$$

$$= x^3+2x^2h+xh^2+$$
$$+x^2h+2xh^2+h^3$$

$$= x^3+3x^2h+3xh^2+h^3$$

$\therefore$ limit above =

$$= \lim_{h\to 0} \frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^2}-2xh-h^2-\cancel{x}-h+\cancel{1}-\cancel{x^3}+\cancel{x^2}+\cancel{x}-\cancel{1}}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3-2xh-h^2-h}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2+3xh+h^2-2x-h-1)}{\cancel{h}}$$

$$= \lim_{h\to 0} 3x^2+3xh+h^2-2x-h-1 \qquad \text{plug } h=0$$

$$= 3x^2+3x(0)+0^2-2x-0-1 = 3x^2-2x-1.$$

$\therefore$ for $f(x) = x^3-x^2-x+1$, $f'(x) = 3x^2-2x-1$

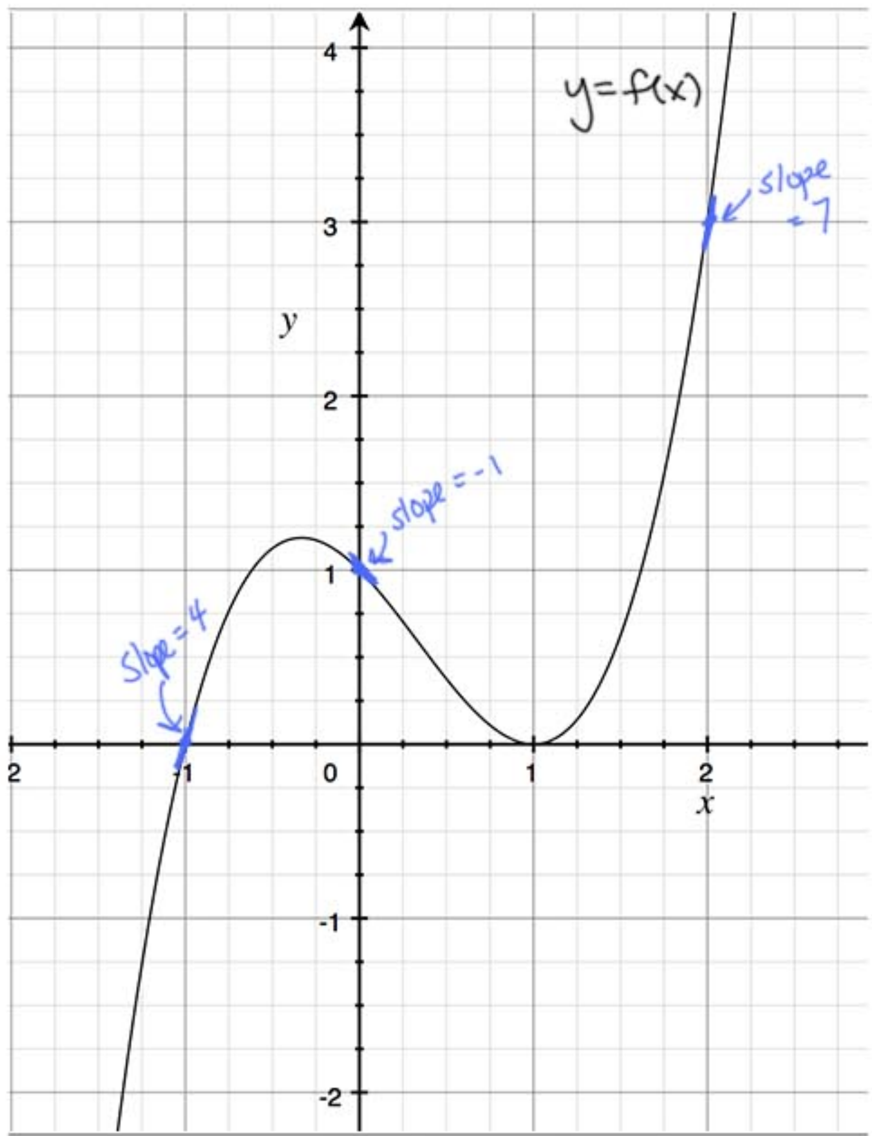
slope function
for $f(x)$.

$\therefore$ slope at $x=-1$ is $f'(-1) = 3(-1)^2 - 2(-1) - 1$
$= 3 + 2 - 1 = 4.$

slope at $x=0$ is $f'(0) = 3(0)^2 - 2(0) - 1$
$= -1$

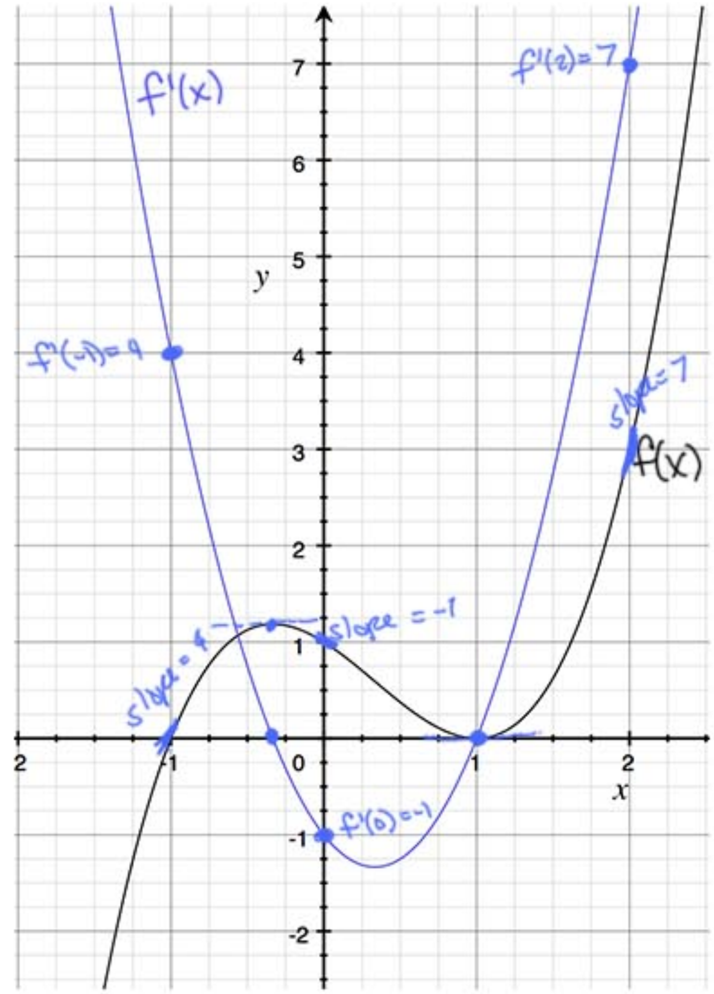
slope at $x=2$ is $f'(2) = 3(2)^2 - 2(2) - 1$
$= 3(4) - 4 - 1 = 7.$

$\therefore$ using the derivative function $f'(x)$ allows us to find the slope of f at multiple points with only one limit computation.



Let's compare the graphs of $f(x)$ and $f'(x)$:

the slope of $f(x)$ gives the y-values for the graph of $f'(x)$.



Ex. Find $f'(x)$ using the definition of the derivative:

$$f(x) = 4 - \sqrt{x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{4 - \sqrt{x+h} - (4 - \sqrt{x})}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{h(\sqrt{x} + \sqrt{x+h})} = \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{x} + \sqrt{x+h}} = \frac{-1}{\sqrt{x} + \sqrt{x}} = \frac{-1}{2\sqrt{x}}$$

$\therefore$ for $f(x) = 4 - \sqrt{x}$, $f'(x) = \dfrac{-1}{2\sqrt{x}}$.

Ex. Find $g'(x)$ using the definition of the derivative:

$$g(x) = \frac{x}{x-2}$$

Work on this problem on your own

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{\frac{x+h}{x+h-2} - \frac{x}{x-2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x+h}{x+h-2} \cdot \frac{x-2}{x-2} - \frac{x}{x-2} \cdot \frac{x+h-2}{x+h-2}}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)(x-2) - x(x+h-2)}{(x+h-2)(x-2)h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + \cancel{hx} - \cancel{2x} - 2h - \cancel{x^2} - \cancel{xh} + \cancel{2x}}{(x+h-2)(x-2)h}$$

$$= \lim_{h \to 0} \frac{-2\cancel{h}}{(x+h-2)(x-2)\cancel{h}} = \lim_{h \to 0} \frac{-2}{(x+h-2)(x-2)}$$

$$= \frac{-2}{(x-2)(x-2)} = \frac{-2}{(x-2)^2} = g'(x).$$

Derivative Notation:

For $y = f(x)$, we can denote the derivative function by:

$$f'(x) \qquad y' \qquad \frac{dy}{dx} \qquad \frac{df}{dx}$$

note: $\frac{d}{dx}(\ )$ = "the derivative of ( ) with respect to x"

When evaluating at $x=a$, we write

$$f'(a) \qquad y'(a) \qquad \left.\frac{dy}{dx}\right|_{x=a} \qquad \left.\frac{df}{dx}\right|_{x=a}.$$

Ex. Above we found for $f(x) = 4-\sqrt{x}$,

$$f'(x) = \frac{-1}{2\sqrt{x}}.$$

We can write $\quad y' = \frac{-1}{2\sqrt{x}} \quad \frac{dy}{dx} = \frac{-1}{2\sqrt{x}} \quad \frac{df}{dx} = \frac{-1}{2\sqrt{x}}.$

And evaluating at $x=1$, for example, we have

$$f'(1) = -\frac{1}{2} \qquad y'(1) = -\frac{1}{2} \qquad \left.\frac{dy}{dx}\right|_{x=1} = -\frac{1}{2} \qquad \left.\frac{df}{dx}\right|_{x=1} = -\frac{1}{2}.$$

Def. A function $f(x)$ is <u>differentiable</u> at $x=a$ if $f'(a)$ exists. $f(x)$ is differentiable on the interval (a,b) if $f'(x)$ exists for every $x \in (a,b)$.

When does $f'(a)$ not exist?

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

this can fail to exist when:

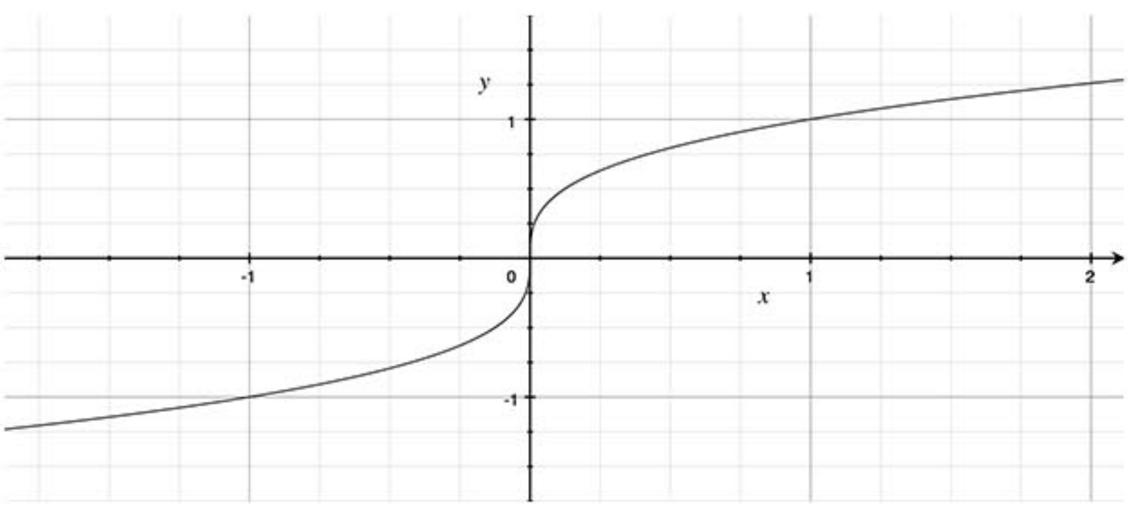
the limit is infinite (vertical tangent line)

the left and right limits are not equal

- f has a corner point at $x = a$
- f has a discontinuity at $x = a$.

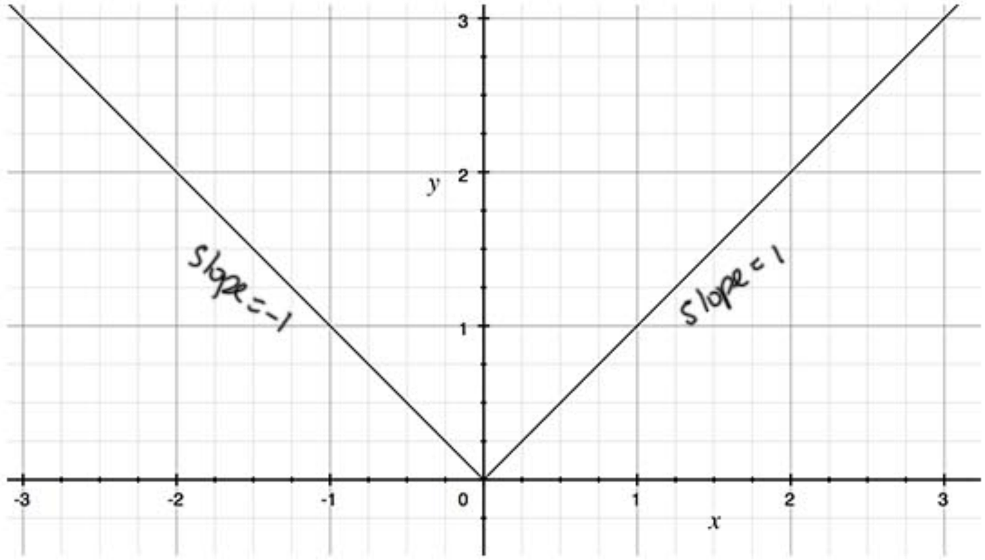
Ex. $f(x) = x^{1/3}$ at $x = 0$, The slope is infinite.

So $f'(0)$ DNE, and f is not differentiable at $x = 0$.



f is differentiable on $(-\infty, 0) \cup (0, \infty)$.

Ex. $f(x) = |x|$



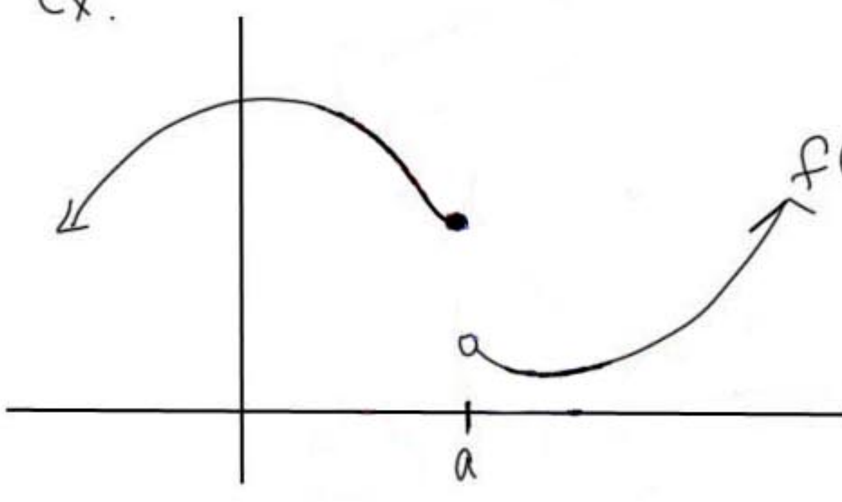
$f'(0)$ DNE

f is not differentiable at $x = 0$

since $\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} \neq \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$.

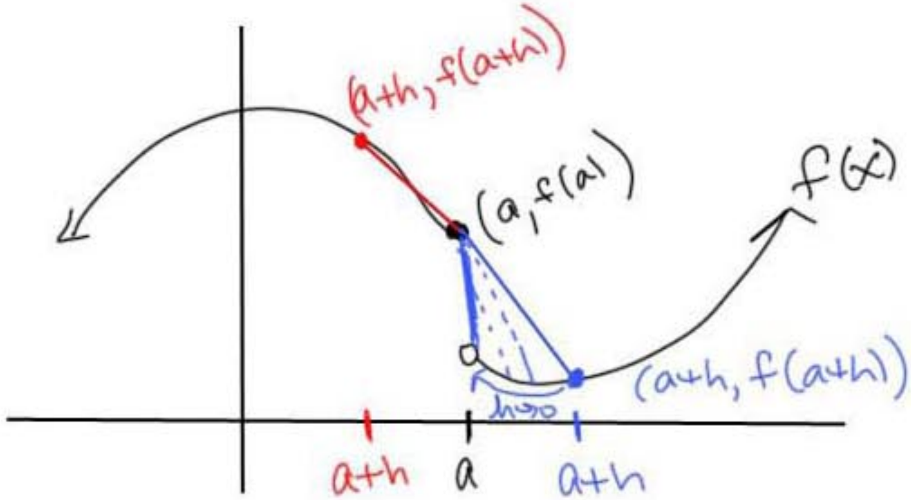
f is differentiable on $(-\infty, 0) \cup (0, \infty)$.

Ex.



f is not continuous at $x = a$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



$$\lim_{\substack{h \to 0^- \\ h<0}} \frac{f(a+h)-f(a)}{h} = \text{finite}$$

$$\lim_{\substack{h \to 0^+ \\ h>0}} \frac{f(a+h)-f(a)}{h} = \text{infinite} \Rightarrow$$

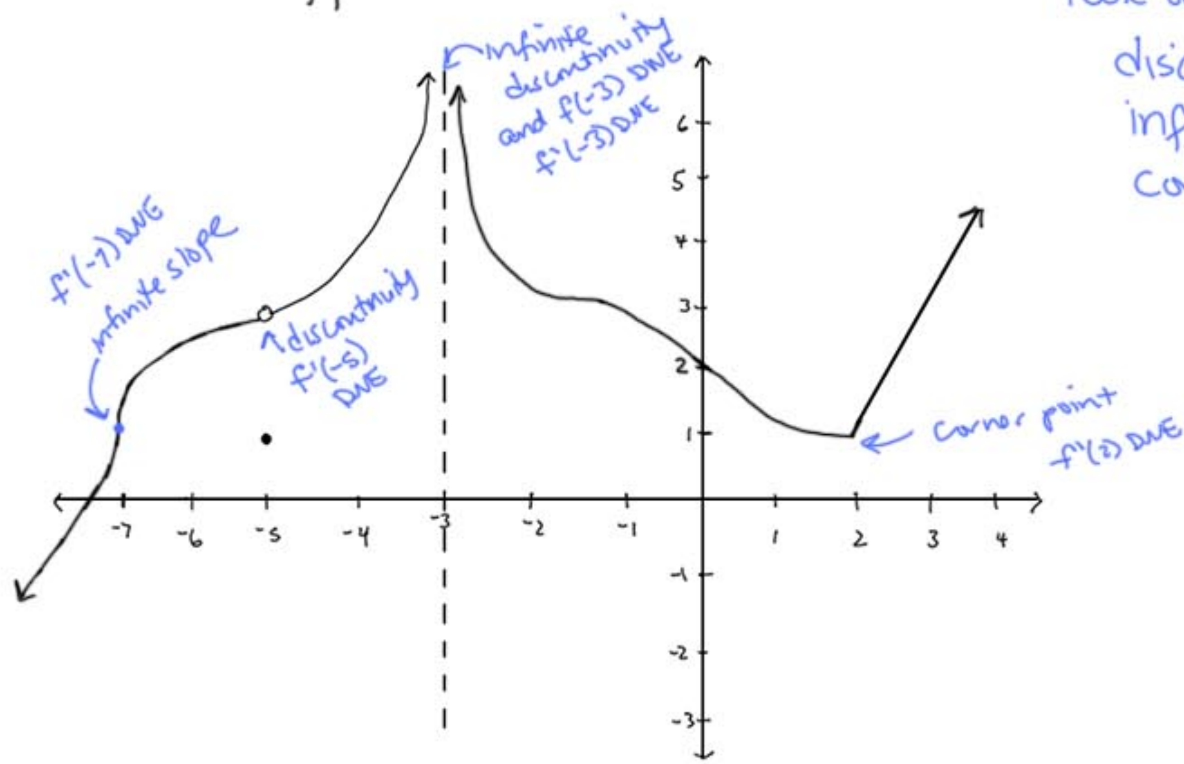
$f'(a)$ DNE as a real #.

$\therefore$ differentiability implies continuity.

Theorem: If f is differentiable at $x=a$, then f is continuous at $x=a$.

Ex. State the intervals on which the function is differentiable.

look out for:
discontinuities
infinite slopes
corner points



$(-\infty, -7) \cup (-7, -5) \cup (-5, -3) \cup (-3, 2) \cup (2, \infty)$ f is differentiable

State the intervals on which the above function is continuous: $(-\infty, -5) \cup (-5, -3) \cup (-3, \infty)$

Higher Order Derivatives

If $f'(x)$ is also a differentiable function, we can take its derivative : $f''(x)$ ("f double prime of x")

$f''(x)$ is the slope function for $f'(x)$.

But as compared to the original $f(x)$, $f''(x)$ offers the rate of change of the slope of f, i.e. the curvature, or concavity of f. (more on this in later lessons)

If $f(t)$ is a distance function,
$f'(t)$ is the corresponding velocity function,
and $f''(t)$ is the corresponding acceleration function.

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Ex. above we saw for $f(x) = x^3 - x^2 - x + 1$

$$f'(x) = 3x^2 - 2x - 1 .$$

then $f''(x) = \lim_{h \to 0} \dfrac{f'(x+h) - f'(x)}{h}$

$$= \lim_{h \to 0} \frac{3(x+h)^2 - 2(x+h) - 1 - (3x^2 - 2x - 1)}{h}$$

$$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - 2x - 2h - 1 - 3x^2 + 2x + 1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{-2x} - 2h \cancel{-1} \cancel{-3x^2} \cancel{+2x} \cancel{+1}}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h} = \lim_{h \to 0} \frac{\cancel{h}(6x + 3h - 2)}{\cancel{h}}$$

$$= \lim_{h \to 0} 6x + 3h - 2 = 6x + 3(0) - 2 =$$

$$= 6x - 2 = f''(x).$$

$\therefore$ for $f(x) = x^3 - x^2 - x + 1$

$$f'(x) = 3x^2 - 2x - 1$$

$$f''(x) = 6x - 2 .$$

Notation for higher order derivatives:

we said above for $y = f(x)$ we have

first derivative: $f'(x) \quad y' \quad \frac{dy}{dx} \quad \frac{df}{dx}$

Now, second derivative: $f''(x) \quad y'' \quad \frac{d^2y}{dx^2} \quad \frac{d^2f}{dx^2}$.

We could also have a third derivative:

$$f'''(x) \quad y''' \quad \frac{d^3y}{dx^3} \quad \frac{d^3f}{dx^3}$$

"f triple prime of x"

In general, the n^{th} derivative

$$f^{(n)}(x) \quad y^{(n)} \quad \frac{d^ny}{dx^n} \quad \frac{d^nf}{dx^n}$$

So $f^{(2)}(x) = f''(x)$ different than $f^2(x) = (f(x))^2$

$y^{(3)} = y'''$