# Math 20100
# Calculus I

# Lesson 16
# Maximum and Minimum Values
# (Absolute Extrema)

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# Maximum and Minimum Values
(Absolute Extrema)

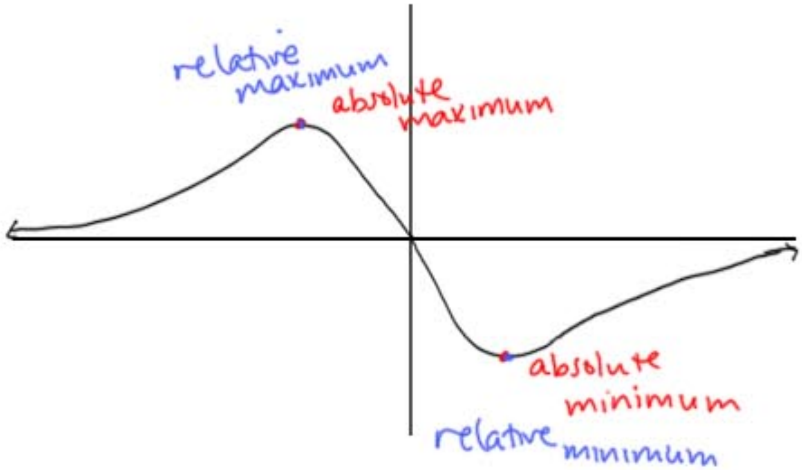
<u>Def</u>. A function $f$ has an <u>absolute maximum</u> (or <u>global maximum</u>) at $x = c$ if $f(c) \geq f(x)$ $\forall x \in D$ domain of $f$.

for all (for $\forall$), in (for $\in$)

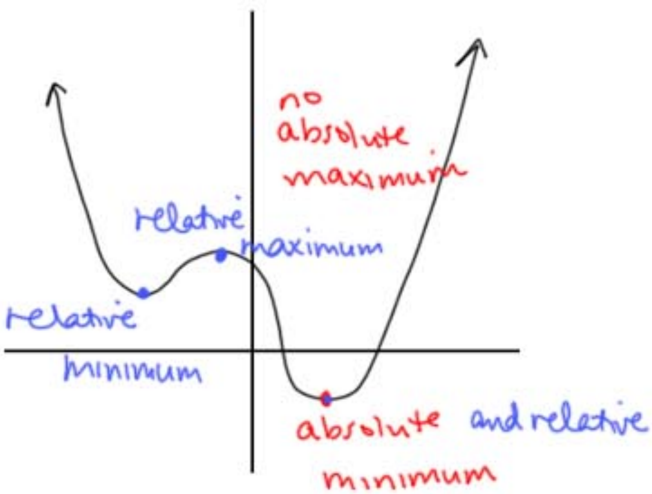
$f(c)$ is the (absolute) maximum value of $f$.

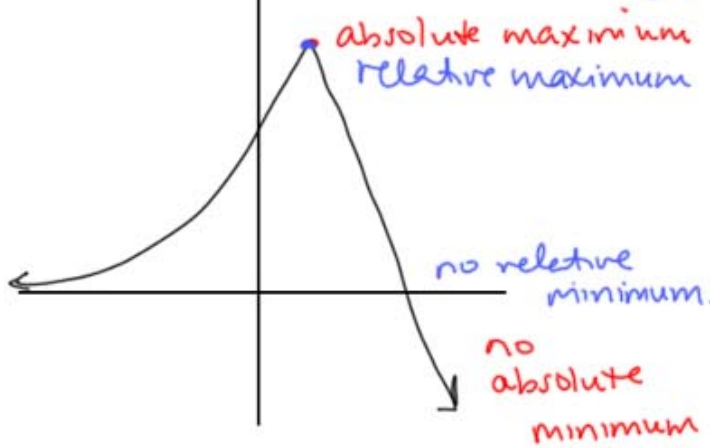
<u>Def</u>. A function $f$ has an <u>absolute minimum</u> (or <u>global minimum</u>) at $x = c$ if $f(c) \leq f(x)$ $\forall x \in D$ domain of $f$.

$f(c)$ is the (absolute) minimum value of $f$.








Def. A function $f$ has a <u>relative maximum</u> (or <u>local maximum</u>) at $x = c$ if

$$f(c) \geq f(x) \quad \forall x \text{ in an open interval containing } x = c.$$

Def: A function $f$ has a <u>relative minimum</u> (or <u>local minimum</u>) at $x = c$ if

$$f(c) \leq f(x) \quad \forall x \text{ in an open interval containing } x = c.$$

Ex. Graph the function and identify the absolute and relative extrema:

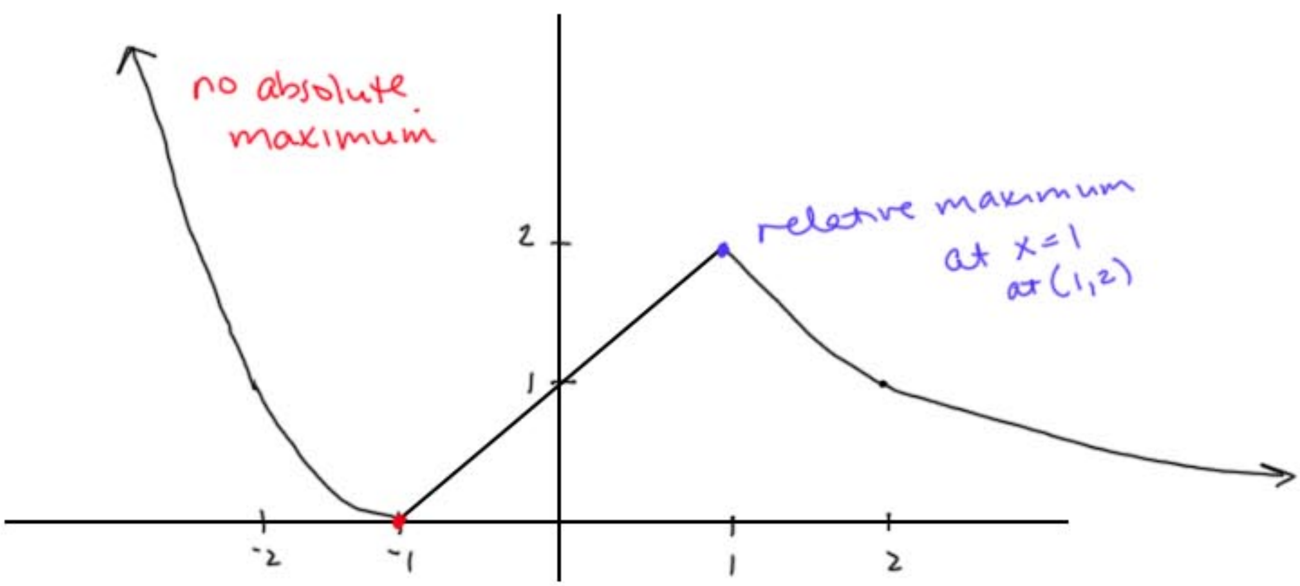
$$f(x) = \begin{cases} (x+1)^2 & x \le -1 \\ x+1 & -1 < x < 1 \\ \frac{2}{x} & x \ge 1 \end{cases}$$

parabola left 1 unit

line with slope = 1, intercept = 1

$y = \frac{1}{x}$


Work on this problem on your own




absolute minimum value of 0 when x=-1

also a relative minimum

Extrema can occur:

where $f'(x) = 0$ (slope = 0)

where $f'(x)$ DNE

at endpoints of the domain

Def $x = c$ is a critical number of $f$

if: $\left.\begin{array}{l} f'(c) = 0 \\ \text{or } f'(c) \text{ DNE} \end{array}\right\}$ c must be in the domain of f.

Call point $(c, f(c))$ a critical point of f.

Ex. Find the critical numbers of $f(x) = x^3 + x^2 - x$.

Start by finding $f'(x)$. $f'(x) = 3x^2 + 2x - 1$.

Notice $f'(x)$ always exists. So any critical numbers will come from $f'(x) = 0$.

$3x^2 + 2x - 1 = 0$

$(3x-1)(x+1) = 0$ $\Rightarrow$ critical numbers of f are $x = \frac{1}{3}$, $x = -1$.

$3x - 1 = 0 \quad x + 1 = 0$

$x = \frac{1}{3} \quad x = -1$

Ex. Find the critical points of $f(x) = 3x^3 - 9x + 5$

**STOP** Work on this problem on your own

$f'(x) = 9x^2 - 9$ always exists, set $= 0$

$$9x^2 - 9 = 0$$

$$9x^2 = 9$$

$x^2 = 1 \qquad x = \pm 1$ critical numbers

for critical points, need the corresponding y-values as well.

$$f(x) = 3x^3 - 9x + 5$$

$$\left.\begin{array}{ll} f(1) = 3(1)^3 - 9(1) + 5 = 3 - 9 + 5 = -1 & (1, -1) \\ f(-1) = 3(-1)^3 - 9(-1) + 5 = -3 + 9 + 5 = 11 & (-1, 11) \end{array}\right\} \text{critical points}$$

Ex. Find the critical numbers for $f(x) = x^{1/3}$.

$$f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}}.$$

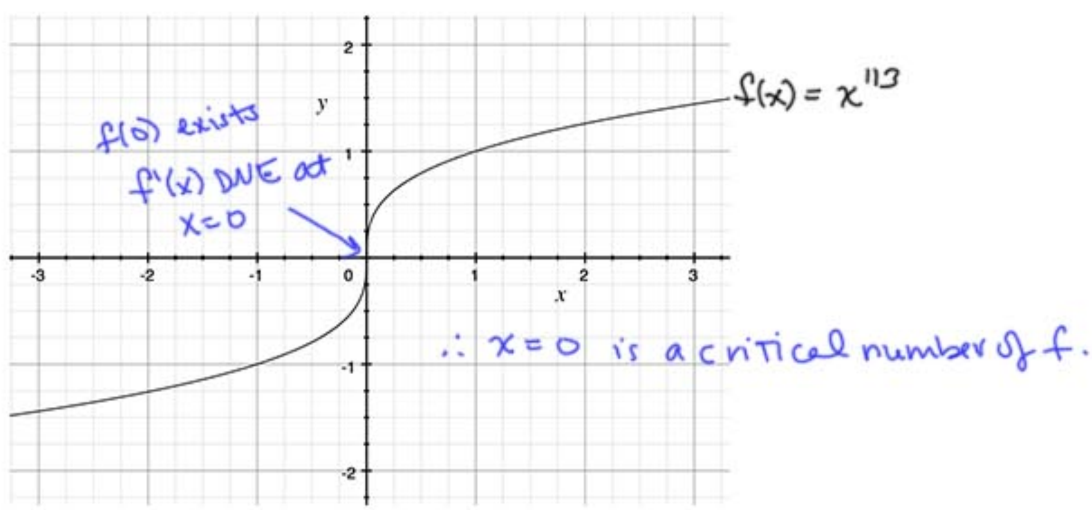
Notice $\frac{1}{3x^{2/3}}$ can never $= 0$.

(fraction $= 0$ means numerator $= 0$,
denominator $\neq 0$)

Also notice $\frac{1}{3x^{2/3}}$ does not exist at $x = 0$

<u>and</u> $x = 0$ is in the domain of $f$.

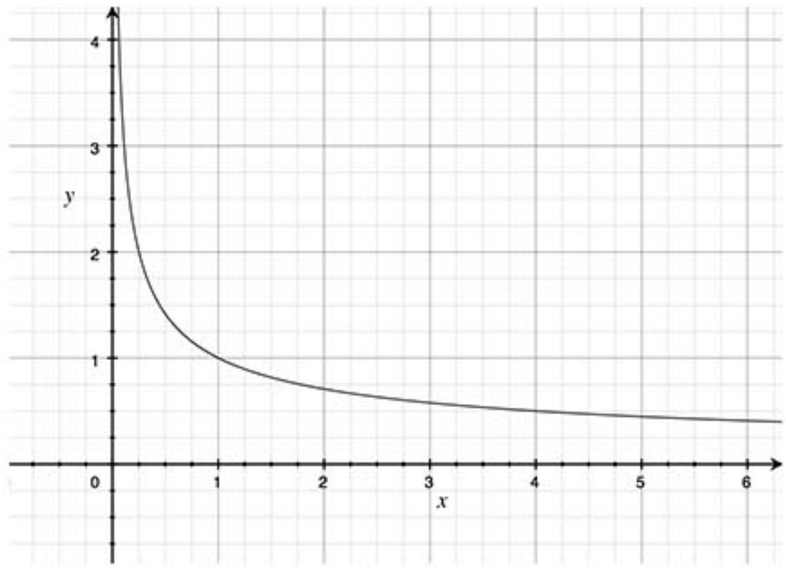
$\therefore$ $x = 0$ is a critical number of $f(x) = x^{1/3}$.




Ex. Find any critical numbers of $g(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$

$g'(x) = -\frac{1}{2}x^{-3/2} = -\frac{1}{2x^{3/2}}$ never $= 0$.

DNE at $x = 0$. <u>BUT</u> $x = 0$ is <u>not</u> in the domain of $g$. So $x=0$ would not be considered a critical number. And then here, There are no critical numbers.




$g(x) = \frac{1}{\sqrt{x}}$

Ex. Find the critical points: $f(x) = \dfrac{x^2}{(2x+1)^3}$

$$f'(x) = \frac{(2x+1)^3(2x) - (x^2)\,3(2x+1)^2(2)}{(2x+1)^6}$$

$f'(x)$ Does not exist where $2x+1=0$, but that $x$-value is not in the domain anyway.

$f'(x)=0$ where $(2x+1)^3(2x) - (x^2)(3)(2x+1)^2(2) = 0$

factor: $2x(2x+1)^2\left[2x+1-3x\right] = 0$

$$2x(2x+1)^2(1-x)=0$$

$$2x=0 \qquad \underbrace{2x+1=0}_{\text{not in domain}} \qquad 1-x=0$$

$$x=0 \qquad\qquad\qquad x=1$$

$f(0)=0 \qquad f(1)=\frac{1}{3^3}=\frac{1}{27}$

$(0,0)$ and $(1,\frac{1}{27})$ are the critical points.

Ex. $g(t)=|3t-4|$ find the critical numbers

$$g(t)=|3t-4|=\begin{cases}3t-4 & 3t-4\geq 0\\ -(3t-4) & 3t-4<0\end{cases}$$

$$|x|=\begin{cases}x & x\geq 0\\ -x & x<0\end{cases} \qquad =\begin{cases}\underline{3t-4} & \underline{t\geq \frac{4}{3}}\\ \underline{-3t+4} & \underline{t<\frac{4}{3}}\end{cases}$$

and $g'(t)=\begin{cases}3 & t>4/3\\ -3 & t<4/3\end{cases}$

note, $g'(4/3)$ DNE. ($g$ is continuous at $t=4/3$, but not diffble.)

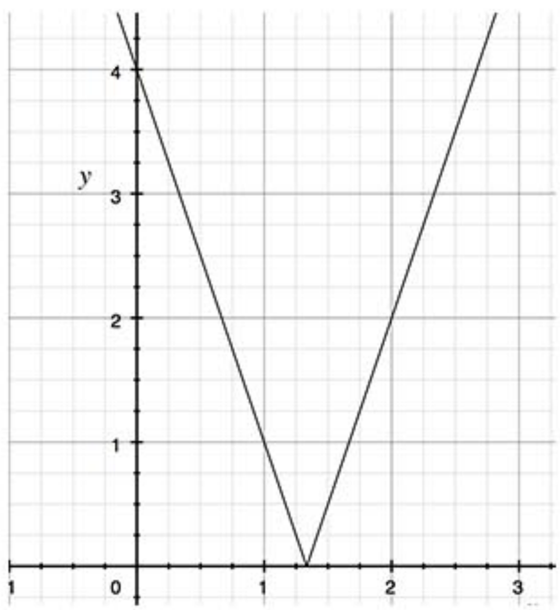
so $t = 4/3$ is a critical number.

also need to find the $t$-values for which

$$g'(t) = 0 \quad \text{but} \quad g'(t) = \begin{cases} 3 & t > 4/3 \\ -3 & t < 4/3 \end{cases}$$

no $t$-values such that $g'(t) = 0$.

so the only critical number is $t = \frac{4}{3}$.



Ex. find the critical numbers of

$$f(x) = |-x^2+4|$$

$$= \begin{cases} -x^2+4 & -x^2+4 \geq 0 \Rightarrow x^2 \leq 4 \quad -2 \leq x \leq 2 \\ -(-x^2+4) & -x^2+4 < 0 \quad x^2 > 4 \Rightarrow x > 2 \text{ or } x < -2 \end{cases}$$

$$f(x) = \begin{cases} -x^2+4 & -2 \leq x \leq 2 \\ x^2-4 & x < -2 \text{ or } x > 2 \end{cases}$$

continuous at $x = \pm 2$

so far

$$f'(x) = \begin{cases} -2x & -2 < x < 2 \\ 2x & x < -2 \text{ or } x > 2 \end{cases}$$

we're not yet sure if $f'(2)$, $f'(-2)$ exist

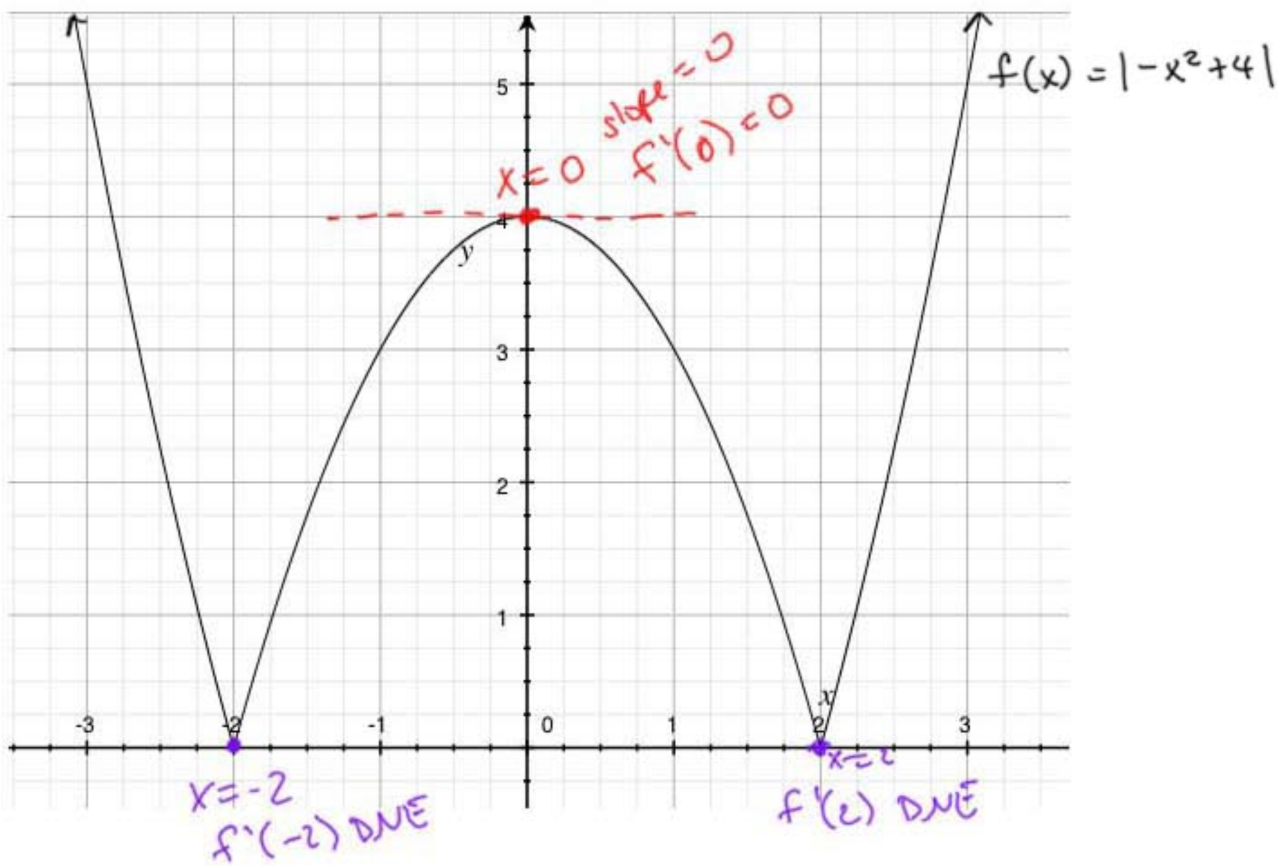
since we have continuity at $x = \pm 2$, we can check $f'$ from the left + right at $x = \pm 2$

at $x = -2$, left slope $(x < -2)$ $2(-2) = -4$; right slope $(x > -2)$ $-2(-2) = 4$ $\Big\}$ $f'(-2)$ DNE

$$\text{at } x=2, \quad \left.\begin{array}{ll} \text{left slope } (x<2) & -2(2) = -4 \\ \text{right slope } (x>2) & 2(2) = 4 \end{array}\right\} f'(2) \text{ DNE}$$

$\therefore$ $x = \pm 2$ are critical numbers of $f$.

also $f'(x) = 0$ when $x = 0$, so $x = 0$ is also a critical number of $f$.



Ex. Find the critical numbers of $f(x) = 2\cos x + \cos^2 x$.

$$f'(x) = 2(-\sin x) + 2\cos x(-\sin x)$$

$$= -2\sin x - 2\sin x \cos x$$

$$= -2\sin x(1+\cos x) \overset{\text{set}}{=} 0$$ note $f'(x)$ exists $\forall x$

$$-2\sin x = 0 \qquad 1+\cos x = 0$$

$$\qquad\qquad\qquad -1 \qquad\quad -1$$

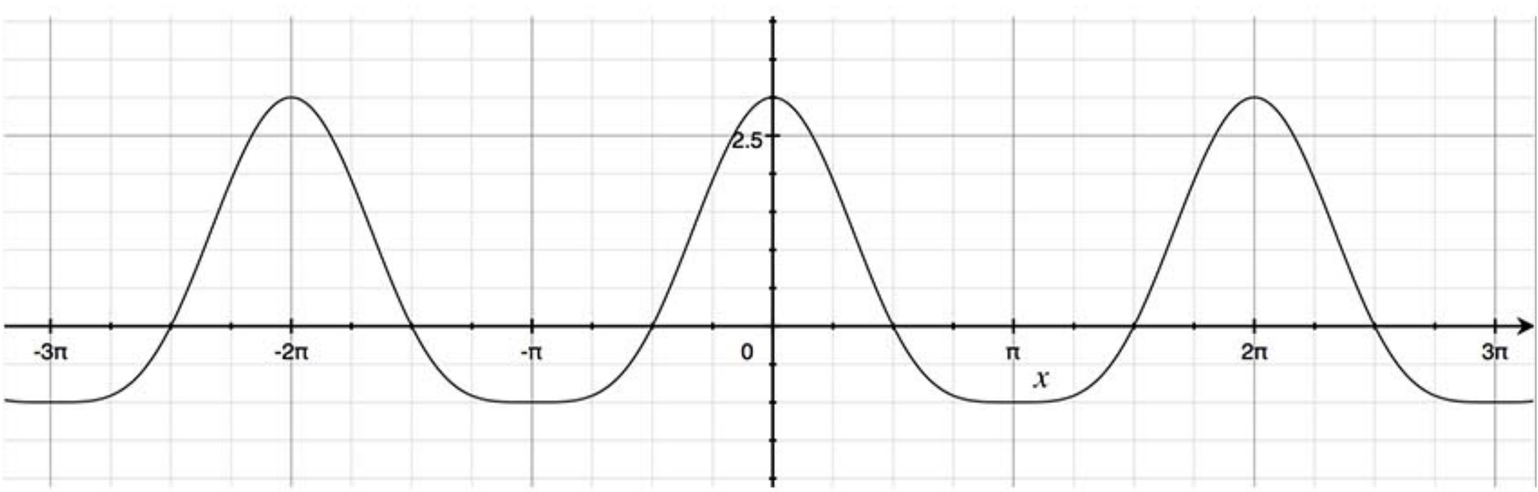
$$\sin x = 0 \qquad \cos x = -1$$

$$x = k\pi \quad k \in \mathbb{Z} \qquad x = (2k+1)\pi \quad k \in \mathbb{Z}$$

integer multiples of $\pi$ odd multiples of $\pi$

integer multiples of $\pi$

solution: $x = k\pi$ for $k \in \mathbb{Z}$.



Ex. Find the critical numbers of

$$g(x) = \sec^2 x - 2\tan x.$$


Work on this problem on your own

$$g'(x) = 2\sec x \cdot \sec x \tan x - 2\sec^2 x$$

$$= 2\sec^2 x \tan x - 2\sec^2 x$$

$$= 2\sec^2 x(\tan x - 1) \overset{\text{set}}{=} 0$$

note $g'(x)$ DNE when $\cos x = 0$ (below)

$2\sec^2 x = 0$ $\qquad$ $\tan x - 1 = 0$

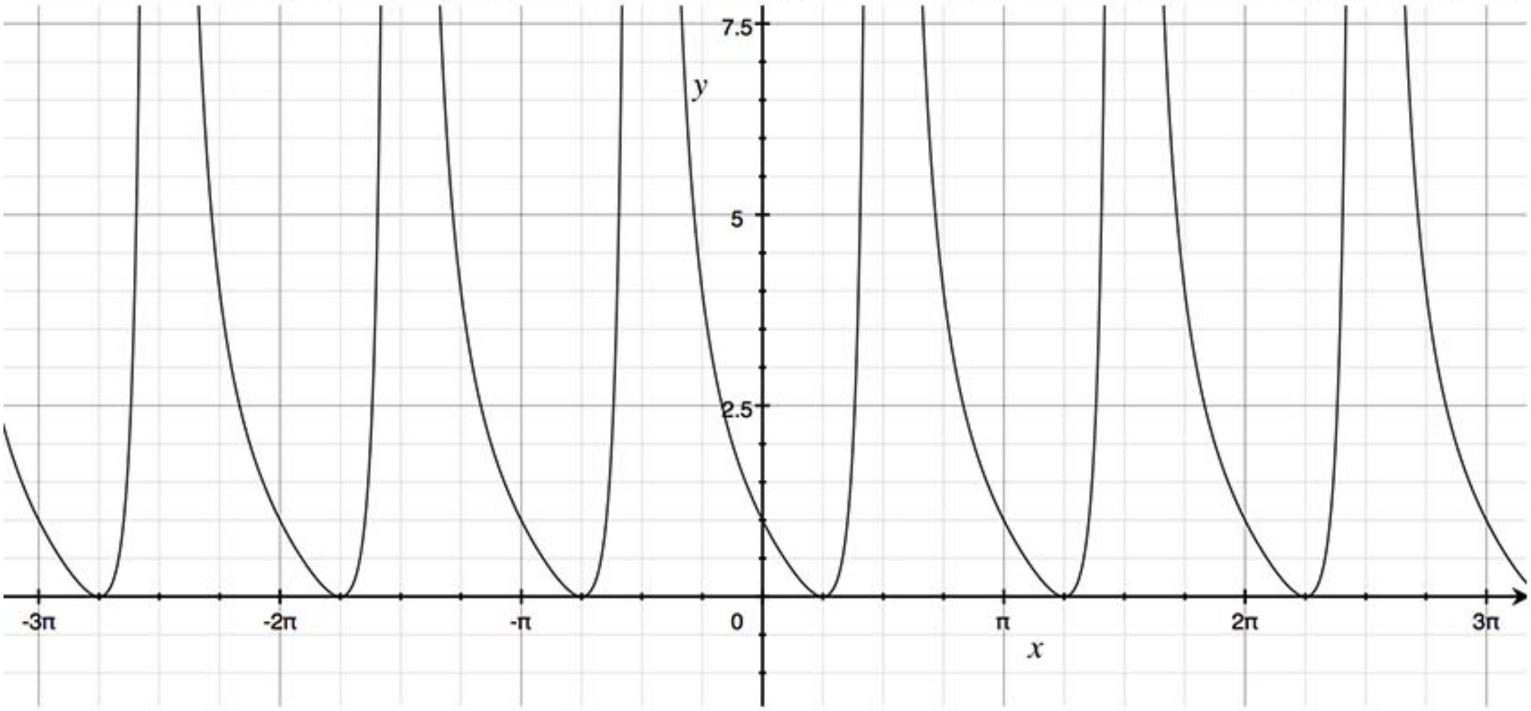
$\dfrac{2}{\cos^2 x} = 0$ $\qquad$ $\tan x = 1$

no solution

$$\left.x = \frac{\pi}{4} + \pi k \text{ for } k \in \mathbb{Z}\right\} \text{critical numbers}$$

$g'(x)$ DNE when $\cos x = 0$ $\quad$ ($g(x)$ DNE)

$$\left.\begin{aligned} x &= (2k+1)\cdot\frac{\pi}{2} \quad k \in \mathbb{Z} \\ &\text{odd multiples of } \frac{\pi}{2} \\ &= \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z} \end{aligned}\right\} \text{not critical numbers}$$




Summary: to find critical points of $f(x)$:

1) find $f'(x)$

2) ~~set~~ $f'(x) = 0$ + solve for $x$

3) find the $x$-values for which $f'(x)$ DNE

   a) fractions, zero in denominator

   b) piecewise, check the $x$-values for which the function changes

# Extreme Value Theorem:

if $f$ is continuous on $[a,b]$ closed interval

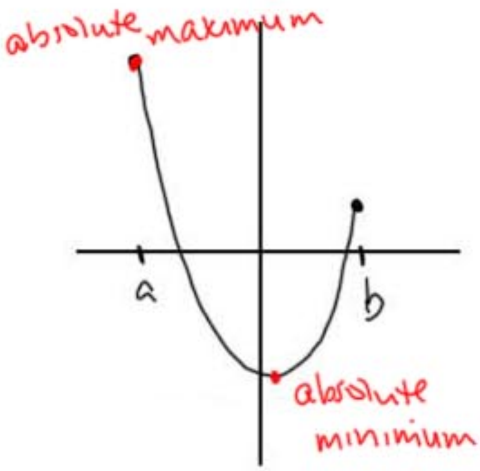
then $\exists$ an absolute maximum on $[a,b]$

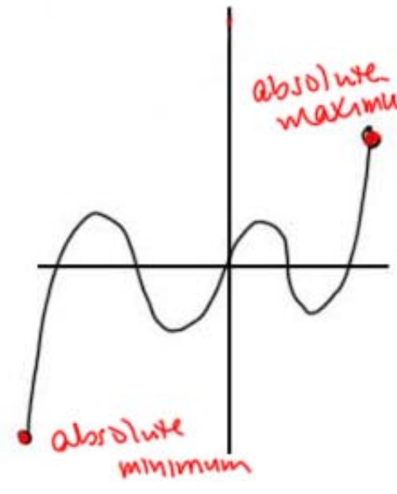
and an absolute minimum on $[a,b]$.

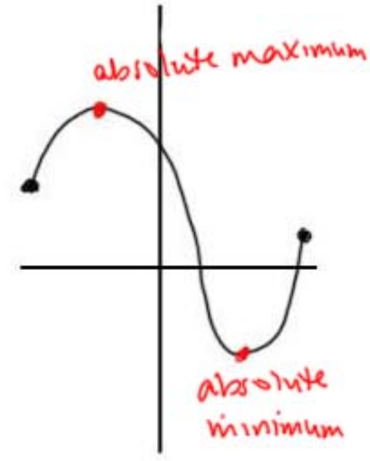
there exists

When we are given a closed interval $[a,b]$ as the domain, There will be endpoints to The graph.

And continuous on $[a,b]$ means no jumps, holes, or asymptotes.



Ex. find the absolute max and absolute min

$$f(x) = 3x^2 - 12x + 5 \quad \text{over } [0,3]$$

steps to finding absolute extrema:

① find critical numbers of f in given interval

② plug critical numbers and endpoints into original function to see which x-values have the highest / lowest y-values.

Ex. ① $f'(x) = 6x - 12 = 0$

$6x = 12 \Rightarrow x = 2$

(f' always exists)  yes $2 \in [0,3]$

②

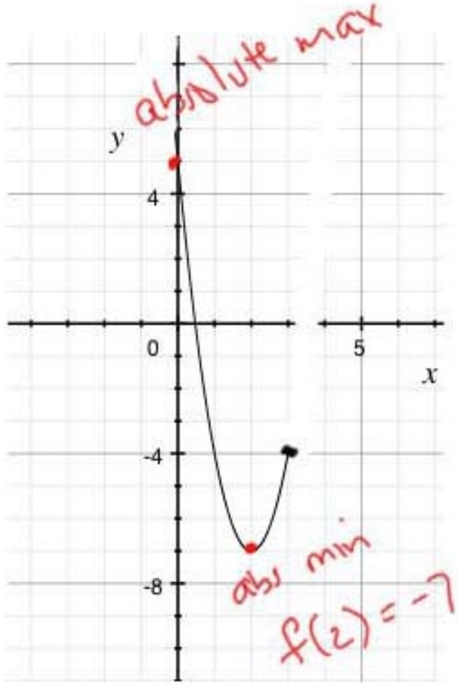
| $x$ | $f(x) = 3x^2 - 12x + 5$ |
|---|---|
| 0 | 5 |
| 2 | -7 |
| 3 | -4 |

endpoints of interval: 0, 3

critical point: 2

highest y-value: absolute max at $x = 0$, maximum value of $f$ is $= 5$.

lowest y-value: absolute min at $x = 2$, minimum value $= -7$.




$f(0) = 5$

Ex. Find the absolute extrema of

$f(x) = 2\cos x + \cos^2 x$ on $\left[-\frac{\pi}{2}, \pi\right]$.

We saw above that $f$ has critical numbers at integer multiples of $\pi$.

So in $\left[-\frac{\pi}{2}, \pi\right]$, critical numbers at $x = 0, x = \pi$.

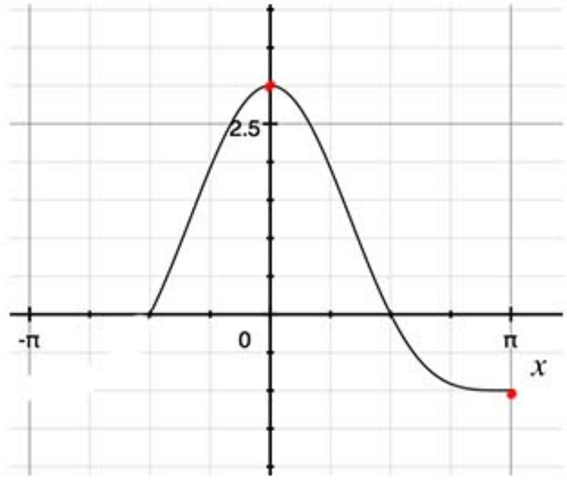
| $x$ | $f(x) = 2\cos x + \cos^2 x$ |
|---|---|
| $-\frac{\pi}{2}$ | $2\cos(-\frac{\pi}{2}) + \cos^2(-\frac{\pi}{2}) = 2(0) + 0 = 0$ |
| $0$ | $2\cos(0) + \cos^2(0) = 2(1) + 1 = 3$ ← highest |
| $\pi$ | $2\cos\pi + \cos^2\pi = 2(-1) + (-1)^2 = -1$ ← lowest |

$\therefore$ $f$ has an absolute max at $x = 0$

with an absolute maximum value of 3

$f$ has an absolute min at $x = \pi$

with an absolute minimum value of $-1$




Ex. Find the absolute extrema of $g(x) = \dfrac{x}{x^2+1}$ on $[-1,3]$.


Work on this problem on your own

$$g'(x) = \frac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

notice $g'(x) = 0$ when $1 - x^2 = 0$

$$1 = x^2$$

$x = \pm 1$ ← both are in $[-1, 3]$

$g'(x)$ always exists $(x^2+1 \neq 0)$.

| $x$ | $\frac{x}{x^2+1}$ |
|---|---|
| $-1$ | $-\frac{1}{2}$ |
| $1$ | $\frac{1}{2}$ |
| $3$ | $\frac{3}{10}$ |

← absolute min at $(-1, -1/2)$

← absolute max at $(1, 1/2)$.