

# Math 20100

## Calculus I

### Lesson 29

## Integration by Substitution

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# Integration by Substitution

So far we have the following  
basic integration rules:

$$\int k \, dx = kx + C$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1 \quad (n = -1 \text{ in Calc II})$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\left( \begin{array}{l} \text{also } \int \csc^2 x = -\cot x + C \\ \int \csc x \cot x = -\csc x + C \end{array} \right)$$

But what if our integrand doesn't fit the rules  
function  
inside the  
integral above?

Look for a composition of functions and  
use u-substitution (undoes chain rule)

$$\text{Ex. } \int x (4+x^2)^{10} dx$$

(1) let  $u =$  inside function. we have  
 $4+x^2$  inside  $( )^{10}$ , so  $u = 4+x^2$



Ex.  $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$       ① let  $u = \sqrt{x} = x^{1/2}$

③  $2 \int \sin \sqrt{x} \cdot \underbrace{\frac{1}{2\sqrt{x}} dx}_{du}$       ②  $\begin{cases} u' = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \\ du = \frac{1}{2\sqrt{x}} dx \end{cases}$

$= 2 \int \sin u du$

④  $= 2(-\cos u) + C$

⑤  $= -2 \cos \sqrt{x} + C.$

Ex.  $\int \cos^4 \theta \sin \theta d\theta = \int (\cos \theta)^4 \sin \theta d\theta$

①  $u = \cos \theta$  inside function,  $\cos \theta$  inside  $( )^4$

②  $du = -\sin \theta d\theta$

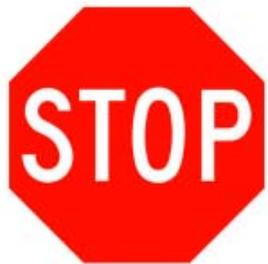
$$\textcircled{3} = - \int \underbrace{(\cos \theta)^4 \sin \theta}_{du} d\theta$$

$$= - \int u^4 du$$

$$\textcircled{4} = - \frac{u^5}{5} + C$$

$$\textcircled{5} = - \frac{\cos^5 \theta}{5} + C$$

Ex.  $\int u \sqrt{1-u^2} du$



Work on this problem  
on your own

let  $w = 1 - u^2$  then  $dw = \underline{-2u du}$

$$-\frac{1}{2} \int \underline{-2u} \sqrt{1-u^2} \underline{du} = -\frac{1}{2} \int \sqrt{w} dw$$

$$= -\frac{1}{2} \int w^{1/2} dw = -\frac{1}{2} \frac{w^{3/2}}{\frac{3}{2}} + C$$

$$= -\frac{1}{2} \cdot \frac{2}{3} w^{3/2} + C$$

$$= -\frac{1}{3} (1-u^2)^{3/2} + C.$$

Definite Integrals with u-sub:

Ex.  $\int_0^{\sqrt{\pi}} 3x \sin(x^2) dx$

let  $u = x^2$

then  $du = 2x dx$

$$= \frac{3}{2} \int_0^{\sqrt{\pi}} \underbrace{2x \sin(x^2)}_{du} dx$$

$$= \frac{3}{2} \int_0^{\pi} \sin u \, du$$

when  $x=0$ ,  $u=0^2=0$   
when  $x=\sqrt{\pi}$ ,  $u=(\sqrt{\pi})^2=\pi$

\* the bounds must match the variable.

$$= \frac{3}{2} (-\cos u) \Big|_0^{\pi}$$

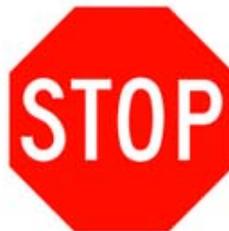
\* don't go back to  $x$ !  
move forward with  $u$ .

$$= \frac{3}{2} [(-\cos \pi) - (-\cos 0)]$$

$$= \frac{3}{2} [(-(-1)) - (-1)]$$

$$= \frac{3}{2} [1+1] = 3.$$

Ex.  $\int_1^3 \cos\left(\frac{\pi t}{2}\right) dt$



Work on this problem  
on your own

$$u = \frac{\pi t}{2} \text{ then } du = \underline{\frac{\pi}{2}} dt$$

$$\frac{2}{\pi} \int_1^3 \frac{\pi}{2} \cos\left(\frac{\pi t}{2}\right) dt = \frac{2}{\pi} \int_{\pi/2}^{3\pi/2} \cos u du$$

$$\text{when } t=1, u = \frac{\pi(1)}{2} = \frac{\pi}{2}$$

$$t=3, u = \frac{\pi(3)}{2} = \frac{3\pi}{2}$$

$$= \frac{2}{\pi} \sin u \Big|_{\pi/2}^{3\pi/2} = \frac{2}{\pi} \left[ \left( \sin \frac{3\pi}{2} \right) - \left( \sin \frac{\pi}{2} \right) \right]$$

$$= \frac{2}{\pi} [-1 - 1] = \underline{\underline{-\frac{4}{\pi}}}$$

Sometimes The substitution requires solving for  $x$  in terms of  $u$ :

$$\text{Ex. } - \int \frac{x^2}{\sqrt{1-x}} dx \quad u = 1-x \Rightarrow x = 1-u$$

$$\underline{du = -dx}$$

$$= - \int \frac{(1-u)^2}{u^{1/2}} du = - \int \frac{1-2u+u^2}{u^{1/2}} du$$

$$= - \int \left( \frac{1}{u^{1/2}} - \frac{2u}{u^{1/2}} + \frac{u^2}{u^{1/2}} \right) du$$

$$= - \int \left( u^{-1/2} - 2u^{1/2} + u^{3/2} \right) du$$

$$= - \left( 2u^{1/2} - 2 \cdot \frac{2}{3} u^{3/2} + \frac{2}{5} u^{5/2} \right) + C$$

$$= -2u^{1/2} + \frac{4}{3} u^{3/2} - \frac{2}{5} u^{5/2} + C$$

$$= -2(1-x)^{1/2} + \frac{4}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2} + C.$$

Sometimes we have a double substitution:

$$\text{Ex. } \int \frac{\sin \sqrt{x}}{\sqrt{x} \cos^5 \sqrt{x}} dx$$

let  $u =$  The innermost function

$$= \int \frac{\sin \sqrt{x}}{\sqrt{x} (\cos \sqrt{x})^5} dx$$

$$u = \sqrt{x} = x^{1/2}$$

$$\text{then } du = \frac{1}{2} x^{-1/2} dx$$

$$= \frac{1}{2\sqrt{x}} dx$$

$$= 2 \int \frac{\sin \sqrt{x}}{2\sqrt{x} (\cos \sqrt{x})^5} \frac{dx}{du}$$

$$= 2 \int \frac{\sin u}{(\cos u)^5} du$$

still a composition...

$$\text{let } w = \cos u$$

$$\text{then } dw = -\sin u du$$

$$= -2 \int \frac{-\sin u}{(\cos u)^5} du$$

$$= -2 \int \frac{dw}{w^5} = -2 \int w^{-5} dw = -2 \frac{w^{-4}}{-4} + C$$

$$= \frac{1}{2w^4} + C = \frac{1}{2(\cos u)^4} + C$$

$$= \frac{1}{2(\cos \sqrt{x})^4} + C$$

$$\text{or} = \frac{1}{2 \cos^4 \sqrt{x}} + C.$$