COMPUTING FLUX OF A VECTOR FIELD $\vec{F}$ ACROSS AN ORIENTED SURFACE S.

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General Definition

The formula

$$\text{flux} = \iint_S \vec{F} \cdot \vec{n} \mathrm{d}\sigma = \iint_R \vec{F} \cdot (\pm \vec{r}_u \times \vec{r}_v) \mathrm{d}u \mathrm{d}v.$$

works for computing any flux integral of a vector field across an oriented surface. It is all you need to know. However the formula is error prone when used on an exam since you need to compute the cross product of the partials $\vec{r}_u \times \vec{r}_v$. Many students prefer to memorize a few of the common forms of $\vec{n}\mathrm{d}\sigma$ that show up over an over on exams. Before you blindly memorize the formulas below, you should check that they can be derived from the formula above.

To understand the formula above you should know

$$\vec{n}\mathrm{d}\sigma = \pm \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| \mathrm{d}u \mathrm{d}v = \pm \vec{r}_u \times \vec{r}_v \mathrm{d}u \mathrm{d}v$$

and $\mathrm{d}\sigma = |\vec{r}_u \times \vec{r}_v| \mathrm{d}u \mathrm{d}v$ for surface S parametrized by $\vec{r}(u, v)$. R is the region in the uv-plane which is mapped to S by the parametrization $\vec{r}(u, v)$. Ultimately, if you do integrate, you will integrate over the region R as a double integral.

Graphs

A graph $z = f(x, y)$ can be parametrized by $\vec{r}(x, y) = x\vec{i} + y\vec{j} + f(x, y)\vec{k}$. In this case

$$\vec{n}\mathrm{d}\sigma = \pm \langle -f_x, -f_y, 1 \rangle \mathrm{d}x \mathrm{d}y$$

and $\mathrm{d}\sigma = \sqrt{(f_x^2 + f_y^2 + 1)} \mathrm{d}x \mathrm{d}y$. The region of integration, R, is determined by the shadow in the xy-plane determined by the surface S.

Spheres

The sphere centered at the origin with radius a is given by the equation $x^2 + y^2 + z^2 = a^2$. It can be parametrized by $\vec{r}(\phi, \theta) = a \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$. Then

$$\vec{n} = \pm \frac{1}{a} \langle x, y, z \rangle$$

and $\mathrm{d}\sigma = a^2 \sin\phi \mathrm{d}\phi \mathrm{d}\theta$. The region of integration, R, is determined by noting that θ is the longitude angle and is bounded by $0 \leq \theta \leq 2\pi$ to give a full rotation around the sphere. ϕ is the latitude angle and is measured from 0, the North pole, to π, the South pole. When $\phi = \frac{\pi}{2}$ and θ goes from 0 to 2π you will parametrize the equator.

Date: March 29, 2016.

Cylinders

The cylinder centered around the z-axis with radius a is given by the equation $x^2 + y^2 = a^2$. It can be parametrized by $\vec{r}(\theta, z) = a\langle \cos\theta, \sin\theta, z\rangle$. Then

$$\vec{n} = \pm\frac{1}{a}\langle x, y, 0\rangle$$

and $\mathrm{d}\sigma = a\mathrm{d}\theta\mathrm{d}z$. The region of integration is determined by noting that z is the height and θ is the angle around the z-axis.

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