9.2 Geodesic Equations

Theorem 9.2.1 : A curve γ on a surface S is a geodesic iff for any part $\gamma(t) = \sigma(u(t), v(t))$ of γ satisfies the following two second order nonlinear ODE:

- $u''E + \frac{1}{2}u'^2 E_u + u'v'E_v + v''F + (v)'^2 \left(F_v - \frac{1}{2}G_u\right) = 0.$
- $v''G + \frac{1}{2}v'^2 G_v + u'v'G_u + u''F + (u')^2 \left(F_u - \frac{1}{2}E_v\right) = 0.$

9.2 Geodesic Equations

Theorem 9.2.1 : (PROOF) these are the same differential equations as in our textbook but written in a slightly different form.

- We have $\gamma' = u'\sigma_u + v'\sigma_v$
- and then

  $$\gamma'' = u''\sigma_u + u'(u'\sigma_{uu} + v'\sigma_{uv}) + v''\sigma_v + v'(u'\sigma_{uv} + v'\sigma_{vv})$$

  by the chain rule.
- Now the two ODEs arrive by knowing that γ'' must be perpendicular to σ_u and σ_v, i.e. the ODEs are $\gamma'' \cdot \sigma_u = 0$ and $\gamma'' \cdot \sigma_v = 0$.

9.2 Geodesic Equations

The two geodesic equations, like most nonlinear differential equations, are usually difficult or impossible to solve explicitly. However the existence and uniqueness theorem proves there are many solutions of such systems of nonlinear ODE.

- Existence and Uniqueness Theorem for ODE: there is a unique geodesic through any given point of a surface in any given tangent direction.
- The proof of the existence theorem is theoretical. It belongs in an ODE course, but is usually avoided in an undergraduate ODE course.
- The geodesic equations allow us to study geodesics in a patch. We can then use σ to map these plane curves to the surface S.

9.2 Geodesic Equations

EXAMPLE: There is one case in which the geodesic equations are simple to solve: in our favorite surface the Euclidean plane the geodesics are lines. The geodesics in an arbitrary surface S are the analogs of straight lines in the plane. We are studying modern geomtery by following the implications of the geometry determined on the surface S when the "lines" are geodesics. We have already discussed the strange implications of spherical geometry when the "lines" are great circles.

9.2 Geodesic Equations

The following corollary can sometimes be helpful in determining geodesics of a surface S.

- Corollary 9.2.7 : Any local isometry f between surfaces

  $$f : S_1 \to S_2$$

  takes geodesics γ_1 of S_1 to geodesics $\gamma_2 = f \circ \gamma_1$ of S_2.
- Proof: since the surfaces are locally isometric they have identical first fundamental forms:

  $$E_1 = E_2, F_1 = F_2, G_1 = G_2$$

  and therefore the geodesic ODEs are identically solved for γ_1 and $\gamma_2 = f \circ \gamma_1$.

9.2 Geodesic Equations

Here are two clever ways to use Corollary 9.2.7 to show that the geodesics of the sphere $x^2 + y^2 + z^2 = a^2$ are great circles.

- Method 1 : verify that the great circles through the North Pole satisfy the geodesic ODE with $I = du^2 + \cos u dv^2$ and then rotate.
- Proof by contradiction: Assume the geodesic γ_1 that starts at P and ends at Q on the sphere and does not follow the great circle determined by P and Q. Then the mirror reflection through the great circle through P and Q would give a second geodesic γ_2 through P and Q which contradicts the uniqueness part of the ODE.

9.2 Geodesic Equations

Here is a clever idea using Corollary 9.2.7 to show that the geodesics of the cylinder $x^2 + y^2 = 1$ are helices $\gamma(t) = (\cos t, \sin t, mt + b)$.

- Since we know that the Euclidean plane and the cylinder are isometric, the Corollary demonstrates that all geodesics of the sphere are images of geodesics of the plane, which we know to be lines.

9.2 Geodesic Equations

KEY IDEA: Use the geodesic ODEs in order to study modern geometry in a patch of a surface S. We can then use our understanding of calculus and differential equations to study curves in the plane $(u(t), v(t))$ with different metrics ($=$ measurement sticks $=$ first fundamental form) and different straight lines than the Euclidean plane. The theory that arises from these different aspects is modern geometry.

9.2 Geodesic Equations

- The sphere $x^2 + y^2 + z^2 = 1$ with first fundamental form

$$I = du^2 + f(u)dv^2 = du^2 + \cos^2(u)dv^2.$$

- The implication of this first fundamental form is that geodesics are great circles.
- The area of a spherical triangle is then $\alpha + \beta + \gamma - \pi$
- The parallel postulate is false in Spherical geometry.

9.2 Geodesic Equations

- Spherical geometry is a fascinating example of modern geometry but it is not, for us, the most interesting modern geometry because it fails more than the parallel postulate.
- Spherical geometry fails the postulate that through any two given points P and Q there is a unique line through them, if P and Q satisfy $Q = -P$..
- For us the hyperbolic geometry introduced in Chapter 11 is more interesting. Hyperbolic geometry satisfies all the axioms of Euclidean geometry except the parallel postulate.

9.2 Geodesic Equations

- 2000 year long goal of geometry: Use the first four postulates of Euclidean geometry in order to prove the parallel postulate.
- After many failed attempts mathematicians in the 18th Century turned the problem upside down. They attempted to prove that parallel postulate could not be proved by the other four postulates by exhibiting a geometry where the first four postulates are true but the parallel postulate is false.
- This was completed with the discovery of hyperbolic geometry.

9.3 Geodesic on surfaces of revolution

- Hyperbolic geometry, for us, is motivated by the pseudosphere, the surface of revolution

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$$

  when $f(u) = e^u$ and $g(u) = \int \sqrt{1 - e^{2u}} du$ so that $f'^2 + g'^2 = 1$.
- After reparametrizing by setting $w = e^{-u}$ we get a surface with first fundamental form

$$\frac{dv^2 + du^2}{w^2}$$

9.3 Geodesic on surfaces of revolution

- Forgetting the pseudosphere of revolution we will study the upper-half plane $w > 0$ with metric
$$\frac{dv^2 + du^2}{w^2}.$$
This will be our model of hyperbolic geometry.
- Show that the geodesic ODEs for for hyperbolic geometry reduce to
$$v'' - \frac{2v'w'}{w} = 0$$
$$w'' + \frac{v'^2 - w'^2}{w} = 0.$$
- Show that the lines vertical half-lines $v = v_0, w > 0$ are geodesics in hyperbolic geometry. and the semi-circles
$$(u(t), v(t)) = (u_0 + r\cos\theta(t), r\sin\theta(t)), 0 \leq \theta(t)\pi, r > 0$$
are geodesics of the hyperbolic plane when parametrized by arc length.

9.3 Geodesic on surfaces of revolution

Show that the semi-circles

$$(v(t), w(t)) = (v_0 + r\cos\theta(t), r\sin\theta(t)), 0 \leq \theta(t)\pi, r > 0$$

centered on the v-axis are geodesics of the hyperbolic plane when parametrized by arc length. Hint: first show that $\theta' = k^2 \sin^2\theta$ and $\theta'' = k^2 \sin\theta\cos\theta$ for some constant k because (v, w) is parametrized by arc length

11 Hyperbolic Geometry

We have already learned that spherical geometry resembles Euclidean geometry in some ways but not others. One of the most remarkable discoveries of nineteenth century mathematics is that the pseudosphere has a geometry that more closely resembles Euclidean geometry, with geodesics playing the role of straight lines. In fact the closest correspondence with Euclidean geometry is obtained by embedding the pseudosphere in a larger geometry, which is called *hyperbolic* geometry or *non-Euclidean* geometry.

11 Hyperbolic Geometry

We find that all of the axioms Euclidean geometry hold in hyperbolic geometry, except the parallel postulate which states that if P is a point not on a straight line L then there is a *unique* straight line passing through P that does not intersect L.

- Hyperbolic geometry was discovered independently and almost simultaneously by Bolyai, Lobachevsky, and Gauss in the 1820s.
- The implications of the discovery ended centuries of attempts by Greek, Arab, and later Western mathematicians to deduce the parallel postulate from the other four postulates of Euclidean geometry, and profoundly changed our view of what geometry is.

11.1 Upper half-plane model

We will study hyperbolic geometry in this class by ignoring the pseudosphere and studying an expansion of its surface patch instead.

- For us hyperboic geometry will be the upper half-plane $\mathcal{H} = \{(v, w) \in \mathbb{R}^2 | w > 0\}$ with first fundamental form

$$\frac{dv^2 + dw^2}{w^2}.$$

- It will often be helpful to identify $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$ via

$$(v, w) \to v + iw.$$

11.1 Upper half-plane model

- Proposition 11.1.1 : Hyperbolic angles in $\mathcal{H}$ are the same as Euclidean angles.
- Proposition 11.1.2 : The geodesics in $\mathcal{H}$ are the half-lines parallel to the imaginary axis and the semi-circles with centers on the real axis.
- Proposition 11.1.3 : There is a unique hyperbolic line passing through any two distinct points of $\mathcal{H}$.
- Proposition 11.1.3 : The parallel postulate does not hold in $\mathcal{H}$.

11.1 Upper half-plane model

Since there is a unique hyperbolic line passing through any two points $a, b \in \mathcal{H}$ it makes sense to define the hyperbolic distance $d_{\mathcal{H}}(a, b)$ between points a and b to be the length of the hyperbolic line segment joining them. It can be shown that this hyperbolic length is actually the shortest curve in $\mathcal{H}$ joining a and b.

11.1 Upper half-plane model

- Proposition 11.1.4 : The hyperbolic distance between two points $a, b \in \mathcal{H}$ is

$$d_{\mathcal{H}}(a, b) = 2 \tanh^{-1} \frac{|b - a|}{|b - \bar{a}|}.$$

- The appearance of the hyperbolic tangent gives an indication of the reason $\mathcal{H}$ is called "hyperbolic" geometry.
- Proof 11.1.4 :

$$d = \int_{\phi}^{\psi} \sqrt{\frac{(v')^2 + (w')^2}{w^2}} d\theta = \int_{\phi}^{\psi} \frac{d\theta}{\sin \theta}$$

11.1 Upper half-plane model

- Theorem 11.1.5 : Let $\mathcal{P}$ be an n-sided hyperbolic polygon in $\mathcal{H}$ with internal angles $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then the hyperbolic area of the polygon is

$$\mathcal{A}(\mathcal{P}) = (n-2)\pi - \alpha_1 - \alpha_2 - \ldots - \alpha_n.$$

- In particular for a hyperbolic triangle with angles α, β, γ, the area of the hyperbolic triangle is $\pi - \alpha - \beta - \gamma$.
- This should be compared with the formula $\pi = \alpha + \beta + \gamma$ in Euclidean geometry as well as the formula for the area of the spherical triangle $\alpha + \beta + \gamma - \pi$.

11.2 Isometries of $\mathcal{H}$

In Euclidean geometry two triangles are said to be congruent if one triangle can be moved until it coincides with the other. The types of motion that are allowed are combinations of rotations, translations, and reflections, i.e. the isometries of the plane (see Appendix 1). Similarly in Section 6.5 we found that the isometries of the sphere were one, two, or three combinations of mirror reflections over the great circles.

11.2 Isometries of $\mathcal{H}$

It is easy to identify some isometries of $\mathcal{H}$.

- The Translations: $T_a(z) = z + a, a \in \mathbb{R}$.
- The Reflections: $R_a(z) = 2a - \bar{z}$.
- The Dilations: $D_a(z) = az, a > 0$.
- The Inversion: $\mathcal{I}_{0,1}(z) = \frac{1}{\bar{z}}$.

11.2 Isometries of $\mathcal{H}$

- Proposition 11.2.1 : Any composite of a finite number of the hyperbolic translations, reflections, dilations, and inversions defined in the above slide is an isometry of $\mathcal{H}$.
- In particular the inversion in a circle with center $a \in \mathbb{R}$ on the real axis and radius $r > 0$:

$$\mathcal{I}_{a,r}(z) = a + \frac{r^2}{\bar{z} - a}$$

is an isometry of $\mathcal{H}$.

11.2 Isometries of $\mathcal{H}$

- Proposition 11.2.2 : The inversion $\mathcal{I}_{a,r}$ in the circle with the center $a \in \mathbb{R}$ and radius $r > 0$ takes hyperbolic lines that intersect the real axis perpendicularly at a to half-lines, and all other hyperbolic lines to semicircles.
- Proposition 11.2.3 : Let l_1 and l_2 be hyperbolic lines and z_1 and z_2 points on l_1 and l_2 respectively. Then there is an isometry of $\mathcal{H}$ that takes l_1 to l_2 and z_1 to z_2.
- Proposition 11.2.4 : In hyperbolic geometry similar triangles are congruent.