6.5 Spherical geometry

If we are to develop spherical geometry by analogy with Euclidean plane geometry, the first thing we need to do is to decide what should be the analogs of the straight lines. Now straight lines in the plane are the shortest curves joining any two of its points, so it is natural to ask what the corresponding shortest curves are on the sphere? We are going to see that these shortest curves are arcs of great circles.

6.5 Spherical geometry

If p and q are distinct points on S^2 there is always at least one great circle passing through them. If p and q are not antipodal points, i.e. $p \neq -q$ the plane passing through the origin and perpendicular to $p \times -q$ intersects S^2 in a great circle passing through p and q.

6.5 Spherical geometry

Proposition 6.5.1 : Let p and q be distinct points on S^2. If $p \neq -q$ then the shortest great circle arc joining p and q is the unique shortest length joining p and q. If $p = -q$, any great semicircle joining p and q is a shortest curve joining these two points.

6.5 Spherical geometry

Thus the great circles are the spherical analogues of straight lines in the Euclidean plane. One immediate difference between spherical and plane geometry is that there are no parallel lines in spherical geometry, for any two great circles intersect (the two planes containing the two great circles in a diameter of S^2, the endpoints of which are the points of intersection of the two great circles).

6.5 Spherical geometry

- The spherical distance $d_{S^2}(p, q)$ between two points p and q on the unit sphere is given by

$$\cos d_{S^2}(p, q) = p \cdot q.$$

- Proposition 6.5.3 :

$$\cos \gamma = \frac{\cos C - \cos A \cos B}{\sin A \sin B}$$
$$\frac{\sin \alpha}{\sin A} = \frac{\sin \beta}{\sin B} = \frac{\sin \gamma}{\sin C}.$$

6.5 Spherical geometry

The two parts of Proposition 6.5.3 are called, respectively, the "cosine rule" and "sine rule" for spherical triangles. You can see this by substituting the Taylor approximations $\cos A = 1 - \frac{1}{2}A^2$ and $\sin A = A$, etc. into the respective formulas. You will arrive at the "Law of cosines" and the "Law of sines" respectively for Euclidean geometry.

6.5 Spherical geometry

Corollary 6.5.6 :

$$\cos A = \frac{\cos\alpha + \cos\beta\cos\gamma}{\sin\beta\sin\gamma}.$$

This formula is important because it shows that the sides of a spherical triangle are determined by its angles, unlike the situation in plane geometry in which there are similar triangles with the same angles but possibly different sizes.

6.5 Spherical geometry

Much of Euclidean geometry is concerned with the question of when two geometrical figures (such as triangles) are congruent, which means that one figure can be "moved" so that it coincides with the other. The type of motions that are allowed are those that do not change the size or shape of the triangles, namely the isometries of the plane. We need to determine the isometries of the sphere.

6.5 Spherical geometry

Proposition 6.5.7 : Every isomoetry of S^2 is a composite of reflections in planes passing through the origin. In fact, at most three reflections are needed.

7 Curvature of sufaces

We discuss several approaches to the problem of measuring how "curved" a surface is in this chapter. Each approach leads to the same geometric object: the *second fundamental form*. It turns out that a surface is determined up to isometry of $\mathbb{R}^3$ by its first and second fundamental forms, just as a plane curve is determined up to isometry of $\mathbb{R}^2$ by its signed curvature.

7.1 The second fundamental form

As the parameters (u, v) of the patch σ change to $(u + \Delta u, v + \Delta v)$ the surface moves away from the tangent plane at $\sigma(u, v)$ by the distance

$$(\sigma(u + \Delta u, v + \Delta v) - \sigma(u, v)) \cdot \vec{N}.$$

Using the Taylor theorem we compute

$$(\sigma(u+\Delta u, v+\Delta v) - \sigma(u, v)) \cdot \vec{N} = L(\Delta u)^2 + 2M\Delta u \Delta v + N(\Delta v)^2.$$

when $L = \sigma_{uu} \cdot \vec{N}, M = \sigma_{uv} \cdot \vec{N}$, and $N = \sigma_{vv} \cdot \vec{N}$.

7.1 The second fundamental form

- EXAMPLE: Find the first and second fundamental forms of the Euclidean plane $(x, y, 0)$ in $\mathbb{R}^3$.
- EXAMPLE: Find the first and second fundamental forms of the circle $x^2 + y^2 + z^2 = a^2$ in $\mathbb{R}^3$.
- EXAMPLE: Find the first and second fundamental forms of the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

7.2 The Gauss and Weingarten maps

Our second way to define the curvature of an oriented surface is to consider its unit normal vector $\vec{N}$. The way that $\vec{N}$ varies reflects the way in which the surface curves: $\vec{N}$ varies rapidly near a point at which the surface is highly curved and slowly where the surface is slightly curved.

7.2 The Gauss and Weingarten maps

Describe if the following surfaces curve rapidly or slightly.

- EXAMPLE: Find the first and second fundamental forms of the Euclidean plane $(x, y, 0)$ in $\mathbb{R}^3$.
- EXAMPLE: Find the first and second fundamental forms of the circle $x^2 + y^2 + z^2 = a^2$ in $\mathbb{R}^3$.
- EXAMPLE: Find the first and second fundamental forms of the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

7.2 The Gauss and Weingarten maps

- EXAMPLE: Find the Weingarten map of Euclidean plane $(x, y, 0)$ in $\mathbb{R}^3$.
- EXAMPLE: Find the Weingarten map of the circle $x^2 + y^2 + z^2 = a^2$ in $\mathbb{R}^3$.
- EXAMPLE: Find the Weingarten map of the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

7.2 The Gauss and Weingarten maps

The Gauss map $\mathcal{G}$ is the map from the surface S to the unit sphere S^2 that assigns to any point p of S the point $\vec{N}_p$ in S^2. The rate at which $\vec{N}$ varies across S is measured by the derivative of $\mathcal{G}$:

$$D_p\mathcal{G} : T_pS \to T_{\mathcal{G}(p)}S^2.$$

The Weingarten map is defined as $\mathcal{W}_{p,S} = -D_p\mathcal{G}$. (the minus sign will make some future formulas simpler).

7.2 The Gauss and Weingarten maps

Proposition 7.2.2 The second fundamental form is related to the Weingarten map as follows:

$$L du(\vec{v}) du(\vec{w}) + M du(\vec{v}) dv(\vec{w}) + M du(\vec{w}) dv(\vec{v}) + N dv(\vec{v}) dv(\vec{w}) = \langle\langle v, w \rangle\rangle$$

when $\langle\langle v, w \rangle\rangle = \langle \mathcal{W}_{p,S}(v), w \rangle$.

2.6 Inscribed and circumscribed polygons

Definitions 135 :

- If all the vertices of a polygon lie on a circle then polygon is *inscribed* into the circle, and the circle is said to *circumscribe* the polygon.
- If all the sides of a polygon are tangent to a circle then polygon in circumscribed about the circle and the circle is then inscribed into the polygon.

2.6 Inscribed and circumscribed polygons

Theorem 136 :

- About any triangle a circle can be circumscribed and this circle is unique.
- Into any triangle a circle can be inscribed.

2.7 Four concurrency points in a triangle

140 :

- The three perpendicular bisectors to the sides of a triangle intersect at one point (which is the center of the circumscribed circle).
- The three bisectors of the angles of a triangle intersect at one point (which is the center of the inscribed circle)

2.7 Four concurrency points in a triangle

- Theorem 142 : The three medians of a triangle intersect at one point; this point cuts a third part of each median measured from the corresponding side.
- It is known from physics that the intersection point of the medians of a triangle is the *center of mass* (also called the *barycenter*). It always lies inside the triangle.