6.3 Conformal mappings of surfaces

Now that we know how to measure lengths of curves on surfaces, it is natural to ask about angles. Suppose two curves γ and $\bar{\gamma}$ on S intersect at the point p. The angle of intersection θ is defined to be

$$\cos\theta = \frac{\gamma' \cdot \bar{\gamma}'}{\|\gamma'\| \|\bar{\gamma}'\|}$$

6.3 Conformal mappings of surfaces

Find the angle of intersection of the parameter curves
$\gamma(t) = \sigma(t, v_0)$ and $\bar{\gamma}(t) = \sigma(u_0, t)$ at the point $\sigma(u_0, v_0)$.

6.3 Conformal mappings of surfaces

A *conformal* map of surfaces $f : S_1 \to S_2$ is a local diffeomorphism such that if γ_1 and $\bar{\gamma_1}$ are any two curves in S_1 that intersect at $p \in S$ in θ then γ_2 and $\bar{\gamma_2}$, their images under f, intersect at $f(p)$ in θ.

In short, f is conformal if it preserves angles. This is similar to isometries. f is an isometry if f preserves lengths.

6.3 Conformal mappings of surfaces

A local diffeomorphism $f : S_1 \to S_2$ is conformal if and only if

- (Theorem: 6.3.3 :) There is a function λ on S_1 so that

$$f^*\langle v, w \rangle_p = \lambda(p)\langle v, w \rangle_p.$$

- (Corollary 6.3.4) The first fundamental forms of the patches σ of S_1 and $f \circ \sigma$ of S_2 are proportional.

6.3 Conformal mappings of surfaces

- Is our favorite map $f(\theta, \phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$ a conformal map from S_1 the Euclidean plane to the sphere S_2? Is it an isomtery?
- Is our second favorite map $f(\theta, z) = (\cos\theta, \sin\theta, z)$ from S_1 the Euclidean plane to S_2 the cylinder $x^2 + y^2 = 1$ conformal? Is it an isometry?
- Is the map $f(x, y) = (x^2 - y^2, 2xy)$ a conformal map from S_1 the Euclidean plane (without $(0, 0)$) to itself S_2? Is it an isomtery?

6.3 Conformal mappings of surfaces

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Chapter 2: The Circle

- Theorem 104 : Through any three points not lying on the same line, it is possible to draw a circle, and such a circle is unique.
- Theorem 105 : The diameter perpendicular to a chord, bisects the chord.
- Theorem 110 : (Converse Theorems) Congruent chords are equidistant from the center and subtend congruent arcs. Chords equidistant from the center are congruent. Among two non-congruent chords, the one closer to the center is longer.

We see therefore that out of the three possible cases of disposition of a line and a circle, tangency only takes place in the third case, i.e. when the perpendicular to the line dropped from the center is a radius, and in this case the tangency point is the endpoint of the radius lying on the circle.

- Theorem 113 : If a line is perpendicular to the radius at its endpoint lying on the circle, then the line is tangent to the circle. And (vice versa) if a line is tangent to a circle, then the radius drawn to the tangency point is perpendicular to the line.
- Problem 114 : Construct a tangent to a given circle such that it is parallel to the a given line. the chord.