6.4 Equiareal Maps and a Theorem of Archimedes

From calculus III we know that the area $\mathcal{A}_\sigma(R)$ of the part $\sigma(R)$ of a surface path $\sigma : U \to \mathbb{R}^3$ corresponding to a region $R \subset U$ is

$$\begin{aligned}\mathcal{A}_\sigma(R) &= \int_R \|\sigma_u \times \sigma_v\| dudv \\ &= \int_R \sqrt{EG - F^2} dudv.\end{aligned}$$

(EXERCISE: Use this formula to find the area of the upper hemisphere $x^2 + y^2 + z^2 = 9, z \geq 0$ by first finding R.)

6.4 Equiareal Maps and a Theorem of Archimedes

- A local diffeomorphism $f : S_1 \to S_2$ is *equiareal* if it takes any region in S_1 to a region of the same area in S_2
- (Theorem 6.4.5) A local diffeomorphism $f : S_1 \to S_2$ A local diffeomorphism $f : S_1 \to S_2$ is conformal if and only if for any surface patch $\sigma_1(u, v)$ on S_1, the corresponding patch $\sigma_2(u, v) = f \circ \sigma_1(u, v)$ on S_2 satisfies

$$E_1 G_1 - F_1^2 = E_2 G_2 - F_2^2$$

6.4 Equiareal Maps and a Theorem of Archimedes

- EXERCISE: Show that the horizontal projection $f : S_1 \to S_2$ of S_1 the unit sphere $x^2 + y^2 + z^2 = 1$ onto S_2 the cylinder $x^2 + y^2 = 1$ discovered by Archimedes is equiareal.
- EXERCISE: Show that our favorite parametrization $\sigma : S_1 \to S_2$ of part of the Euclidean plane S_1 onto S_2, the unit sphere as $\sigma(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$ is not equiareal.

6.4 Equiareal Maps and a Theorem of Archimedes

Theorem 6.4.7 The area of the spherical triangle on the unit sphere S^2 with internal angles α, β, and γ is

$$\alpha + \beta + \gamma - \pi.$$

6.5 Spherical geometry

If we are to develop spherical geometry by analogy with Euclidean plane geometry, the first thing we need to do is to decide what should be the analogs of the straight lines. Now straight lines in the plane are the shortest curves joining any two of its points, so it is natural to ask what the corresponding shortest curves are on the sphere? We are going to see that these shortest curves are arcs of great circles.

6.5 Spherical geometry

If p and q are distinct points on S^2 there is always at least one great circle passing through them. If p and q are not antipodal points, i.e. $p \neq -q$ the plane passing through the origin and perpendicular to $p \times -q$ intersects S^2 in a great circle passing through p and q.

6.5 Spherical geometry

Proposition 6.5.1 : Let p and q be distinct points on S^2. If $p \neq -q$ then the shortest great circle arc joining p and q is the unique shortest length joining p and q. If $p = -q$, any great semicircle joining p and q is a shortest curve joining these two points.

6.5 Spherical geometry

Thus the great circles are the spherical analogues of straight lines in the Euclidean plane. One immediate difference between spherical and plane geometry is that there are no parallel lines in spherical geometry, for any two great circles intersect (the two planes containing the two great circles in a diameter of S^2, the endpoints of which are the points of intersection of the two great circles).

2 Relative position of a line and circle

- If the distance from the center to a given line is greater than the radius, then the entire line lies outside of the circle.
- If the distance from the center to a given line is less than the radius, then the line intersects the circle in two points and a segment of the line lies inside the circle.
- If the distance from the center to a given line is equal to the radius, then line intersects the circle in one point.

2 Relative position of a line and circle

- If a line is tangent to a circle, then the radius drawn to the tangency point is perpendicular to the line.
- When the perpendicular to a line dropped from the center of a circle is a radius, the tangency point is the endpoint of the radius lying on the circle.

3 Relative position of two circles

- If two circle have a common point A situated outside the line of centers, then they have one more common point A' symmetric to the first point with respect to the line of centers.
- If two circles have a common point situated on the line of centers, then they are tangent to each other.

4 Inscribed and some other angles

Theorem 123: An inscribed angle measures a half of the subtended arc. (an inscribed angle contains as many angular degrees as a half of the arc it intercepts.)

4 Inscribed and some other angles

Corollaries 124 :

- All inscribed arcs intercepting the same arc are congruent.
- Any inscribed angle intercepting a diameter is right.
- Theorem 125 : The angle formed by a chord and a tangent measures half the intercepted arc.

5 Construction problems

- Construct a right triangle given its hypotenuse and one leg.
- Erect a perpendicular to a ray at an endpoint without extending ray.
- Through a given point draw a tangent to a given circle.
- Construct a common tangent to two given circles.

6 Inscribed and circumscribed polygons

Theorem 136 :

- About any triangle, a circle can be circumscribed and such a circle is unique.
- Into any triangle, a circle can be inscribed and such a circle is unique.