Chapter 6: The First Fundamental Form

Perhaps the thing that a bug living on a surface might with to investigate is to measure the distance between two points on the surface. Of course, this distance will usually be different from the distance between these points as measured by an inhabitant of the three dimensional space containing the surface. For such an inhabitant the shortest distance is, as always, along the straight line joining the points.

6.1 Lengths of Curves on Surfaces

- The *first fundamental form* is the dot product

$$\langle v, w \rangle_{p,S} = v \cdot w$$

- If $\gamma(t)$ is a curve lying in the image of the surface patch $\sigma(u, v)$ i.e. $\gamma(t) = \sigma(u(t), v(t))$ we have

$$\langle \gamma', \gamma' \rangle_{p,S} = Eu'^2 + 2Fu'v' + Gv'^2.$$

when $E = \sigma_u \cdot \sigma_u, F = \sigma_u \cdot \sigma_v$, and $G = E = \sigma_v \cdot \sigma_v$.

6.1 Lengths of Curves on Surfaces

- We will often write the first fundamental form using differential notation

$$\langle \gamma', \gamma' \rangle_{p,S} = Edu^2 + 2Fdudv + Gdv^2.$$

- Make sure you understand how to compute the first fundamental form of planes, surfaces of revolution (including the sphere), and the cylinder.

6.2: Isometries of surfaces

The geometric reason that the cylinder $x^2 + y^2 = 1$ and the plane $z = 0$ have the same first fundamental form is that a piece of paper can be "wrapped" onto the cylinder without crumpling the paper. If we draw a curve on the plane γ_1, then after wrapping it becomes a curve on the cylinder γ_2 of the same length. Since the length of both curves are computed as the integral of the square root of the first fundamental form, it is plausible geometrically to think the first fundamental forms of each surface are identical. Experiment suggests that the it is impossible to wrap the plane onto the sphere $x^2 + y^2 + z^2 = 1$ without crumpling so the plane and sphere should not have the same first fundamental form.

6.2 Isometries of surfaces

- A map $f : S_1 \to S_2$ is a *local isometry* if it takes any curve γ_1 to a curve $\gamma_2 = f \circ \gamma_1$ of the same length.
- Theorem 6.2.2: f is a local isometry if the symmetric bilinear forms $\langle , \rangle_p$ and $f^*\langle , \rangle_p$ are equal for all $p \in S$.
- Corollary 6.2.3 : A local diffeomorphism f is a local isometry if and only if the patches σ_1 and $\sigma_2 = f \circ \sigma_2$ have the same first fundamental form for any patch σ_1 of S_1.

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- Theorem 85 : In any parallelogram opposite sides are congruent, opposite angles are congruent, and the sum of angles adjacent to one side is 180°.
- Theorem / Tests 86 : If in a convex quadrilateral: opposite sides are congruent or two opposite sides are equal and parallel, then the quadrilateral is a parallelogram.

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- Theorem 87 : If a quadrilateral is a parallelogram, then its diagonals bisect each other.
- Theorem 87 : In a quadrilateral, if the diagonals bisect, then the quadrilateral is a parallelogram.
- Definition 88 : *Central Symmetry* Two points are symmetric about the point O if O is the midpoint of the line joining the points.

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- In a rectangle the diagonals are congruent. A rectangle has two axes of symmetry.
- Theorem 93 : If on one side of an angle we mark segments congruent to each other and through their endpoints draw parallel lines, then the segments cut on the opposite side of the angle by these parallel lines are also congruent.
- Midline Theorem 95 : The line segment connecting the midpoints of two sides of a triangle is parallel to the third side and congruent to half of it.