1. Find the LU factorization of the following matrices A.
   (a) $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -\frac{7}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.
   (b) $A = LU = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}$.
2. Use the previous LU factorization of the matrices A to solve the systems below. No credit will be given for any other method.
   (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.
   (b) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
3.
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for i in range(1,100):
    L[i,0] = (A[i,0]/A[0,0])
    A[i,:] = A[i,:] - L[i,0]*A[0,:]
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4. Use $PA = LU$ factorization to solve the following systems. No credit will be given for any other method.
   (a) $(x, y, z) = (-1, 1, 1)$
   (b) $(x_1, x_2) = (-2, 1)$
5. (a) $A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}, P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 & 0 \\ .25 & 1 & 0 \\ .5 & -5 & 0 \end{bmatrix}$, and $U = \begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.
   (b) Solve $Ax = b = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$ using the above $PA = LU$ factorization. $(c = (0, 6, 8), x = (-1, 2, 1)$.
6. It is easy to "see" whether a 2×2 matrix is symmetric, i.e. if $A^T = A$. In order to determine if it is positive definite you can verify that its eigenvalues are all positive.
7. (a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$
   (b) $A = \begin{bmatrix} 25 & 5 \\ 5 & 26 \end{bmatrix}, G = \begin{bmatrix} 5 & 0 \\ 1 & 5 \end{bmatrix}$
8. $c_1 = 4, c_2 = -3$ and $x_1 = 7, x_2 = -\frac{3}{2}$.
9. Find the Cholesky factorization, $A = GG^T$, of each of the following symmetric, positive definite matrices using $G = LD^{\frac{1}{2}}$. ANSWER: $G = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}$
10. Given $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$ and $B = \begin{bmatrix} .0001 & 1 \\ 1 & 1 \end{bmatrix}$.
    (a) Of course all such "rough" answers are subjective to the problem at hand. However, matrix A looks nearly singular because its rows are almost parallel. Matrix B does not suffer from being close to singular by computing its determinant is not close to zero.
    (b) $\kappa_2(A) \approx 40002$ and $\kappa_2(B) \approx 2.6184$

11. Find the norms $\|A\|_2 = \lambda_1$ and the condition numbers $\kappa(A) = \frac{\lambda_1}{\lambda_2}$ of the following positive definite matrices (by hand):
   (a) $\|A\|_2 = 100, \kappa(A) = 50$.
   (b) $\|A\|_2 = 3, \kappa(A) = 3$.
   (c) $\|A\|_2 = 2 + \sqrt{2}, \kappa(A) = \frac{2+\sqrt{2}}{2-\sqrt{2}}$.
12. Find the norms $\|A\|_2$ and the condition numbers $\kappa(A)$ of the following positive definite matrices (by hand) from $\sqrt{\lambda_1(A^T A)}$ and $\sqrt{\lambda_n(A^T A)}$:
   (a) $\|A\|_2 = 2, \kappa(A) = 1$.
   (b) $\|A\|_2 = \sqrt{2}, \kappa(A) = \infty$.
   (c) $\|A\|_2 = \sqrt{2}, \kappa(A) = 1$.
13. Given the ill-conditioned matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$.
   (a) $b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $(x = (2, 0))$
   (b) $b = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix}$. $(x = (1, 1))$
   (c) You should see that the solutions, x, to this ill-conditioned problem are very sensitive to small changes in the input b. There is no robust algorithm.
14. Even well conditioned problems can give errors. Given $B = \begin{bmatrix} .0001 & 1 \\ 1 & 1 \end{bmatrix}$.
   (a) $x_1 = 0, x_2 = 1$ with three digit rounding and without pivoting because
   $$\begin{aligned} .0001x_1 + x_2 &= 1 \\ -9999x_2 &= -9998. \end{aligned}$$
   (b) $x_1 = 1, x_2 = 1$ with three digit rounding and with pivoting because
   $$\begin{aligned} x_1 + x_2 &= 2 \\ .9999x_2 &= .9998. \end{aligned}$$
   (c) The actual answer with eight digit rounding is $(x_1, x_2) = (1.00010001, 0.99989999)$.
15. The system $\begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.0001 \end{bmatrix}$ has one solution $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Suppose $\hat{x} = \begin{bmatrix} 3 \\ -0.0001 \end{bmatrix}$ is the approximate solution after running some algorithm.
   (a) $\hat{r} = \begin{bmatrix} 0.0002 \\ 0 \end{bmatrix}$, and $\|\hat{r}\|_\infty = .0002 = \|\hat{r}\|_2$.
   (b) $\frac{\|\hat{r}\|_2}{\|b\|_2} \approx .0000471$.
   (c) $\frac{\|x-\hat{x}\|_2}{\|x\|_2} \approx 1.58$.
   (d) $\frac{||x-\hat{x}||}{||x||} \approx 1.58 \leq 5001 * .0000471 \approx \kappa(A)\frac{||\hat{r}||}{||b||}$.