Name: ______________________________

1. (9 points) Consider the equation $x^4 = x^3 + 10$.
   (a) Find an interval $[a, b]$ of length one inside which the equation has a solution.
   (b) Starting with $[a, b]$ from part (a), how many steps of the Bisection Method are need to approximate the root within 10^{-7}?
   (c) Use Newton's method to approximate a root of the equation by computing x_1 when $x_0 = 1$.

(a) $[2,3]$ or $[-2,-1]$

(b) $\frac{b-a}{2^{n+1}} < \frac{1}{2}10^{-7} \longrightarrow \frac{1}{2^n} < 10^{-7}$ about $n = 25$

since $\frac{1}{2^7} < 10^{-2}$

(c) $f(x) = x^4 - x^3 - 10$

$f'(x) = 4x^3 - 3x^2 \longrightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-10}{1} = 11$

2. (3 points) Find the forward and the backward error for the approximate root $x_a = \frac{5}{4}$ of $f(x) = 3x - 2$.

forward $|x_a - r| = \left|\frac{5}{4} - \frac{2}{3}\right| = \frac{7}{12}$

backward $|f(x_a)| = \frac{7}{4}$

3. (10 points) Given the matrix $A = \begin{bmatrix} 2 & 4 \\ 8 & 17 \end{bmatrix}$.

(a) Find the LU factorization of A.

(b) Solve the system $Ax = b$ when $b = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$ by using the LU decomposition from (a) and then carrying out the two-step back substitution.

(a) $A = \underbrace{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}}_{u}$

(b) $Lc = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$ $c_1 = 2,\ c_2 = 3$

$ux = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $x_2 = 3,\ x_1 = -5$

4. (6 points) In this question you will use Newton's method to approximate the root of $f(x) = x^2$ with initial guess $x_0 = 1$. Find the following approximations

(a) x_1

(b) x_2

(c) x_{25}.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^2}{2x_i} = \frac{1}{2}x_i$$

(a) $1/2$ (b) $1/4 = 1/2^2$ (c) $\frac{1}{2^{25}}$

5. (6 points) Which of the following three fixed-point iterations converge to $\sqrt{5}$? Rank the ones converge from fastest to slowest. Explain.

(a) $x \to \frac{4}{5}x + \frac{1}{x}$ $\to$ (2)

(b) $x \to \frac{x}{2} + \frac{5}{2x}$ $\to$ (1)

(c) $x \to \frac{x+5}{x+2}$.

$\to$ DOES NOT CONV TO $\sqrt{5}$

(a) $g_1(x) = \frac{4}{5}x + \frac{1}{x}$ $\to g_1'(\sqrt{5}) = 3/5 < 1$ converges

(b) $g_2(x) = \frac{x}{2} + \frac{5}{2x}$ $\to g_2'(\sqrt{5}) = 0$ quadratic conv.

6. (6 points) For each of the following, determine whether Fixed-Point Iteration of $g(x)$ is locally convergent to r. Explain.

(a) $g(x) = \frac{(2x-1)}{x^2}, r = 1$

(b) $g(x) = \cos x + \pi + 1, r = \pi$

(c) $g(x) = e^{2x} - 1, r = 0$.

(a) $g'(x) = \frac{2x^2 - 2x(2x-1)}{x^4} = \frac{2x - 2x^2}{x^4} = \frac{2-2x}{x^3}$

$g'(r) = g'(1) = 0$ Loc Conv

(b) $g'(x) = -\sin x$ $\quad g'(\pi) = 0$ Loc Conv

(c) $g'(x) = 2e^{2x}$ $\quad g'(0) = 2$ NOT Loc Conv.