7. Linear Systems: Iterative Methods

Like chapter 5 we solve $Ax = b$ when A is square and nonsingular.

- When A is large, Gaussian elimination is expensive and time consuming.
- Matrix multiplication Av is cheap and fast.

7.1 The need for iterative methods

- Gaussian Elimination introduces fill-in, zeros in L and U where A had a zero. Much fill in is expensive.
- Sometimes we need not solve system exactly.
- Sometimes we have a good approximate guess.
- Sometimes the matrix A is not given but matrix products Av are given.

7.1 The need for iterative methods

Do not let anything above fool you into believing that Gaussian elimination, LU factorization, and Cholesky factorization are not important. They are still the most commonly used methods to solve systems $Ax = b$.

7.1 The need for iterative methods

- Poisson equation: $-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = g(x, y)$ on unit square.
- Dirichlet boundary conditions:
  $u(x, 0) = u(x, 1) = u(0, y) = u(1, y) = 0$.
- Discretizing: $4u_{i,j} - u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} = b_{i,j}$.

7.1 The need for iterative methods

- Express discretized Poisson equation: $Ax = b$ when $n = 3$.
- $A = \begin{bmatrix} J & -I & 0 \\ -I & J & -I \\ 0 & -I & J \end{bmatrix}.$
- $J = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}.$

7.1 The need for iterative methods

- Several methods solve problems like the above $Ax = b$ by transforming it into a fixed point problem.
- Rewrite $Ax = b$ as $0 = f(x) = Ax - b$.
- Transform $0 = f(x)$ to $x = g(x)$.
- Find the FPI sequence $\vec{x}_{k+1} = g(\vec{x}_k)$.

7.2 Stationary iteration and relaxation methods.

- Split $A = M - N$.
- Rewrite $Ax = b$ as $Mx = Nx + b$.
- Rewrite as FPI:
  $x_{k+1} = M^{-1}Nx_k + M^{-1}b = x_k + M^{-1}(b - Ax_k)$.
- $g(x) = x + M^{-1}(b - Ax) = x + M^{-1}r$ when r is the residual.

7.2 Stationary iteration and relaxation methods.

- How to split $A = M - N$? There are many ways. We will study three.
- Jacobi method: $M = D$, the diagonal of A.
- Gauss-Seidel method: $M = E$, the lower triangular part of A.
- Successive over-relaxation (SOR) is a mix of Jacobi and Gauss-Seidel.

7.2 Stationary iteration and relaxation methods.

$r_k = b - Ax_k$ is the residual after the k-th FPI iteration.

- The Jacobi method: $x_{k+1} = x_k + D^{-1} r_k$.
- and then the Gauss-Seidel: $x_{k+1} = x_k + E^{-1} r_k$.
- SOR: $x_{k+1} = x_k + \omega \left((1-\omega)D + \omega E\right)^{-1} r_k$.

7.2 Stationary iteration and relaxation methods.

Example $A = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$. Compute $\vec{x}_1$ and $\vec{x}_2$ when $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for the Jacobi method and then for the Gauss-Seidel method.

7.2 Stationary iteration and relaxation methods.

Example $A = \begin{bmatrix} 7 & 3 & 1 \\ -3 & 10 & 2 \\ 1 & 7 & -15 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$. Compute x_1 when x_0 is the zero vector in three dimensions for the Jacobi method, and then for the Gauss-Seidel method.

7.3 Convergence of Stationary Methods.

Consider the iterative method

$$x_{k+1} = x_k + M^{-1} r_k$$

when r_k is the kth residual. Then the method converges if and only if the spectral radius $\rho(T)$ of the corresponding iteration matrix $T = I - M^{-1}A$ satisfies

$$\rho(T) < 1.$$

The smaller $\|\rho(T)\| < 1$, the faster the convergence.