MATH 201

SPRING 2009 FINAL SOLUTIONS

PART I

$$① \quad y = \frac{x^2+2}{x^2+1} \begin{matrix} \leftarrow f \\ \leftarrow g \end{matrix} \qquad \begin{matrix} f'=2x \\ g'=2x \end{matrix}$$

$$\Rightarrow \quad y' = \frac{gf'-fg'}{g^2} = \frac{(x^2+1)(2x)-(x^2+2)(2x)}{(x^2+1)^2}$$

$$= \frac{2x\left((x^2+1)-(x^2+2)\right)}{(x^2+1)^2} = \frac{2x(-1)}{(x^2+1)^2} = \boxed{\frac{-2x}{(x^2+1)^2}}$$

$$② \quad y = \underbrace{x}_{f}\underbrace{\cos^2(3x)}_{g}$$

$$f'=1$$

$$g' = 2(\cos(3x))^1 \cdot (-\sin(3x))(3) = -6\sin(3x)\cos(3x)$$

$$y' = fg'+gf' = x \cdot (-6\sin(3x)\cos(3x)) + \cos^2(3x) \cdot 1$$

$$= \boxed{-6x\sin(3x)\cos(3x) + \cos^2(3x)}$$

$$③ \quad f(x) = \frac{16}{\sqrt{x}} + 3(2x)^{1/3} = 16x^{-1/2} + 3(2x)^{1/3}$$

$$\Rightarrow f'(x) = -8x^{-3/2} + (2x)^{-2/3}(2) = \frac{-8}{(\sqrt{x})^3} + \frac{2}{(\sqrt[3]{2x})^2}$$

$$\Rightarrow f'(4) = \frac{-8}{8} + \frac{2}{4} = -1 + \frac{1}{2} = \boxed{-\frac{1}{2}}$$

④ $\int \frac{2x^2-3\sqrt{x}+1}{x^2}\,dx = \int (2-3x^{-3/2}+x^{-2})\,dx$

$= 2x - 3\cdot\frac{x^{-1/2}}{-1/2} + \frac{x^{-1}}{-1} + C$

$= 2x + 6x^{-1/2} - x^{-1} + C = \boxed{2x + \frac{6}{\sqrt{x}} - \frac{1}{x} + C}$

⑤ $\int_0^1 \frac{dx}{\sqrt{8x+1}}$ Let $u = 8x+1$

$du = 8\,dx \Rightarrow \frac{du}{8} = dx$

$= \int_1^9 \frac{1}{\sqrt{u}}\cdot\frac{du}{8} = \frac{1}{8}\cdot\frac{u^{1/2}}{1/2}\Big]_1^9 = \frac{1}{4}\sqrt{u}\Big]_1^9 = \frac{1}{4}(3-1) = \boxed{\frac{1}{2}}$

⑥ $\int \cos^4 x \sin x\,dx$

Let $u = \cos x$

$du = -\sin x\,dx \Rightarrow -du = \sin x\,dx$

$-\int u^4\,du = -\frac{u^5}{5} + C = -\frac{(\cos x)^5}{5} + C = \boxed{-\frac{\cos^5 x}{5} + C}$

⑦ a) $f(x) = \sqrt{x} \Rightarrow f(x+h) = \sqrt{x+h}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}\,\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x+0}+\sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$

b) $\underline{x=4}$

$f(4)=\sqrt{4}=2 \Rightarrow$ Point: $(4,2)$

$f'(4)=\frac{1}{2\sqrt{4}}=\frac{1}{4} \Rightarrow m_{tan}=\frac{1}{4}$

$\Rightarrow$ Tangent line is: $y-2=\frac{1}{4}(x-4)$

$\Rightarrow y=\frac{1}{4}x-1+2=\frac{1}{4}x+1$

$\Rightarrow \boxed{y=\frac{1}{4}x+1}$

⑧ a) $\frac{d}{dx}\left[y^3+(x^2-1)y=8\right]$

$3y^2\frac{dy}{dx}+\left((x^2-1)\cdot\frac{dy}{dx}+y(2x)\right)=0$

$\Rightarrow \frac{dy}{dx}(3y^2+x^2-1)=-2xy \Rightarrow \boxed{\frac{dy}{dx}=\frac{-2xy}{3y^2+x^2-1}}$

b) Note: At $\underline{x=1}$, $y^3+((1)^2-1)\cdot y=8 \Rightarrow y^3=8 \Rightarrow y=2$

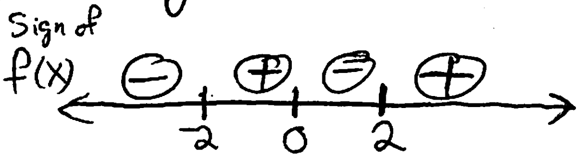
$\Rightarrow$ Point: $(1,2)$

$m_{tan}=\left.\frac{dy}{dx}\right|_{(1,2)}=\frac{-2(1)(2)}{3(2)^2+(1)^2-1}=\frac{-4}{12}=\boxed{-\frac{1}{3}}$

($\Rightarrow$ Tangent line (not requested by problem): $y-2=-\frac{1}{3}(x-1)$)

⑨ $y=x^3-4x \Rightarrow x^3-4x=0 \Rightarrow x(x+2)(x-2)=0$

$\Rightarrow x=0,-2,$ or 2

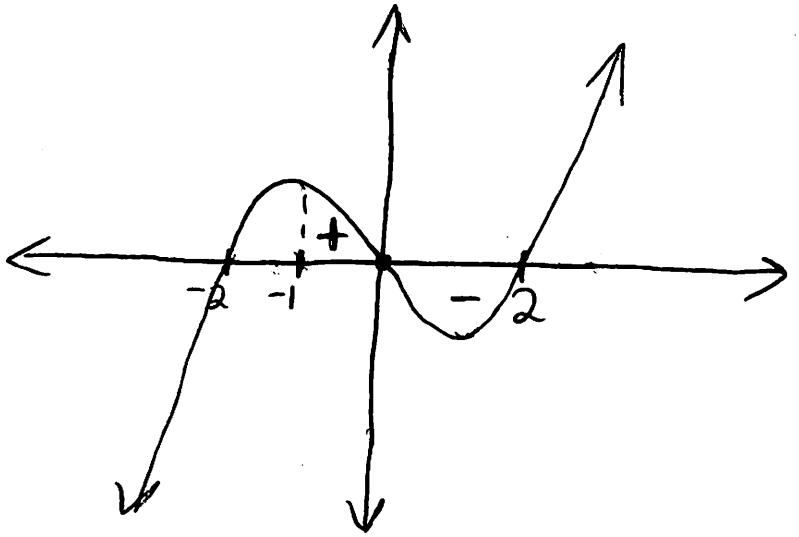


So the area bounded by the x-axis and the graph of $y=x^3-4x$ on $[-1,2]$ is given by: $\int_{-1}^{0}(x^3-4x)dx+-\int_{0}^{2}(x^3-4x)dx$.

$$= \int_{-1}^{0}(x^3-4x)\,dx - \int_{0}^{2}(x^3-4x)\,dx = \left[\frac{x^4}{4}-2x^2\right]_{-1}^{0} - \left[\frac{x^4}{4}-2x^2\right]_{0}^{2}$$

$$= \left[0-\left(\tfrac{1}{4}-2\right)\right] - \left[-4-0\right]$$

$$= 1\tfrac{3}{4} + 4 = \boxed{5\tfrac{3}{4}}$$



⑩ $f(x) = \frac{4x}{x^2+1}$, $f'(x) = \frac{4(1-x^2)}{(1+x^2)^2}$, $f''(x) = \frac{8x(x^2-3)}{(x^2+1)^3}$

A) $\boxed{\text{Intercepts}}$

<u>Y-intercept</u> (x=0)

$$f(0) = \frac{4(0)}{(0)^2+1} = \frac{0}{1} = 0 \Rightarrow (0,0)$$

<u>X-intercept(s)</u> (y=0)

$$\frac{4x}{x^2+1} = 0 \Rightarrow 4x = 0 \Rightarrow x = 0 \Rightarrow (0,0)$$

B) $\boxed{\text{Asymptotes}}$

<u>Vertical</u>

$x^2+1 \neq 0$ for any real number x

$\Rightarrow$ no V.A.'s

<u>Horizontal</u>

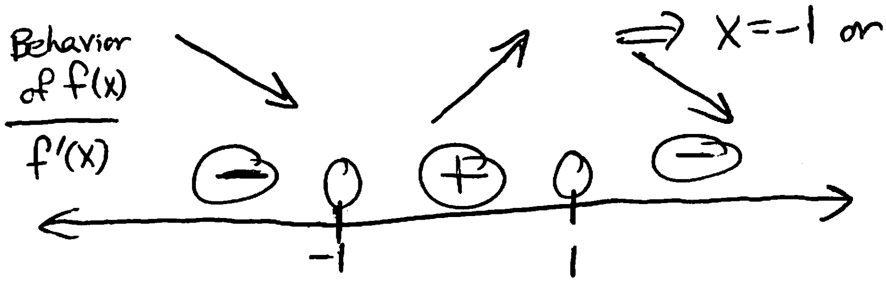
$$\lim_{x\to\infty} \frac{4x}{x^2+1} = 0 \text{ and } \lim_{x\to-\infty} \frac{4x}{x^2+1} = 0$$

$\Rightarrow y = 0$ is an H.A.

C) Local Maxima/Minima

$$f'(x)=0 \Rightarrow 4(1-x^2)=0 \Rightarrow 4(1+x)(1-x)=0$$

$$\Rightarrow x=-1 \text{ or } x=1$$



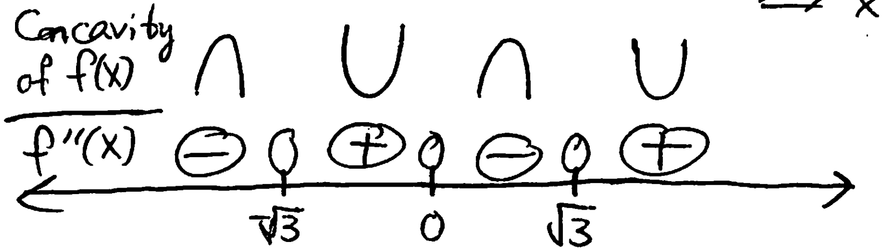
At $x=-1$, $f(x)$ is $\frac{4(-1)}{(-1)^2+1}=\frac{-4}{2}=-2 \Rightarrow (-1,-2)$ is a local min.

At $x=1$, $f(x)$ is $\frac{4(1)}{(1)^2+1}=\frac{4}{2}=2 \Rightarrow (1,2)$ is a local max

D) Concavity/Inflection Point(s)

$$f''(x)=0 \Rightarrow 8x(x^2-3)=0 \Rightarrow x=0 \text{ or } x^2=3$$

$$\Rightarrow x=0 \text{ or } x=\pm\sqrt{3}$$



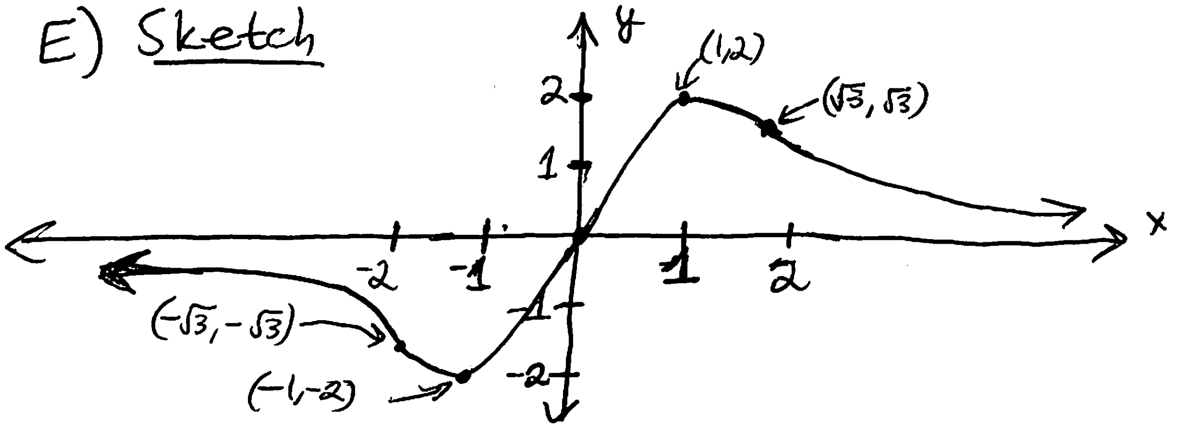
$$f(-\sqrt{3})=\frac{4(-\sqrt{3})}{(-\sqrt{3})^2+1}=-\sqrt{3} \Rightarrow (-\sqrt{3},-\sqrt{3})$$

$$f(0)=\frac{4(0)}{(0)^2+1}=0 \Longrightarrow (0,0)$$

$$f(\sqrt{3})=\frac{4(\sqrt{3})}{(\sqrt{3})^2+1}=\sqrt{3} \Longrightarrow (\sqrt{3},\sqrt{3})$$

} all points of inflection since we have a change in concavity around each of them

E) Sketch



$\boxed{\text{PART II}}$

⑪ a) Let $f(x)=\sqrt{x} \Rightarrow f'(x)=\frac{1}{2\sqrt{x}}$

Know that $\sqrt{9}=3$ & $L_a(x)=f(a)+f'(a)(x-a)$

$\Rightarrow$ Use $a=9$

$f(9)=3$ & $f'(9)=\frac{1}{2(3)}=\frac{1}{6}$

$\Rightarrow L_9(x)=f(9)+f'(9)(x-9)=3+\frac{1}{6}(x-9)$

$\Rightarrow L_9(9.6)=3+\frac{1}{6}(9.6-9)=3+.1=\boxed{3.1}$

b) $f(x)=\cos(2x)$ on $[0,\pi/4]$

$f_{ave}=\frac{1}{b-a}\int_a^b f(x)\,dx$

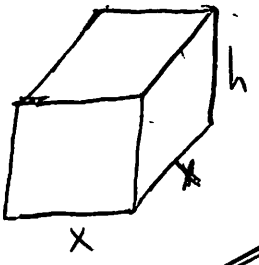
$\Rightarrow f_{ave}=\frac{1}{\pi/4-0}\int_0^{\pi/4}\cos(2x)\,dx=\frac{4}{\pi}\int_0^{\pi/4}\cos(2x)\,dx$

Let $u=2x$

$du=2\,dx \Rightarrow \frac{du}{2}=dx$

$\Rightarrow \frac{4}{\pi}\int_0^{\pi/2}\cos u\cdot\frac{du}{2}\ \frac{4}{\pi}\cdot\frac{1}{2}\sin u\Big]_0^{\pi/2}=\frac{4}{\pi}\frac{1}{2}(1-0)=\frac{4}{\pi}\cdot\frac{1}{2}=\boxed{\frac{2}{\pi}}$

⑫



$V=x^2h=96 \Rightarrow h=\frac{96}{x^2}$

$SA=2x^2+4xh=x^2+x^2+4xh$

~~Cost of SA $=C_{SA}=2(2x^2)+1(4xh)=4x^2+4xh$~~

~~$\Rightarrow C_{SA}=4x^2+4x\left(\frac{96}{x^2}\right)=4x^2+\frac{384}{x}=\frac{4x^3+384}{x}$~~

~~$\Rightarrow C_{SA}'=\frac{x(12x^2)-(4x^3+384)(1)}{x^2}=\frac{12x^3-4x^3-384}{x^2}=\frac{8x^3-384}{x^2}$~~

Cost of \$A $= C_{SA} = \$2(x^2) + \$1(x^2 + 4(xh)$

$$= 2x^2 + x^2 + 4xh = 3x^2 + 4xh$$

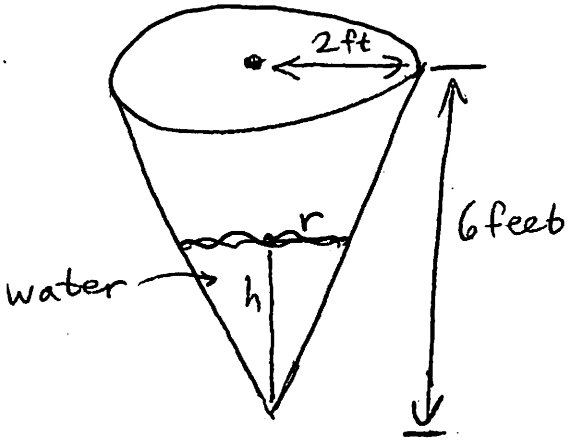
$$\Rightarrow C_{SA} = 3x^2 + 4x\left(\frac{96}{x^2}\right) = 3x^2 + \frac{384}{x} = \frac{3x^3 + 384}{x}$$

$$\Rightarrow C_{SA}' = \frac{x(9x^2) - (3x^3 + 384)(1)}{x^2} = \frac{6x^3 - 384}{x^2} = \frac{6(x^3 - 64)}{x^2}$$

$$\Rightarrow C_{SA}' = 0 \Rightarrow 6(x^3 - 64) = 0 \Rightarrow x^3 = 64 \Rightarrow x = 4$$

$$\boxed{\begin{array}{l} x = 4 \text{ ft} \\ \& \ h = \frac{96}{(4)^2} = 6 \text{ft} \end{array}}$$

⑬



$V = \frac{1}{3}\pi r^2 h$

Given: $\frac{dV}{dt} = -2$ (draining)

Snapshot: $V = 8 \text{ft}^3$

(Should) Know: $\frac{r}{h} = \frac{2 \text{ feet}}{6 \text{ feet}} = \frac{1}{3}$

(at any point in time)

$\Rightarrow h = 3r$

$\Rightarrow \frac{dh}{dt} = 3\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{3}\frac{dh}{dt}$

Want: $\frac{dh}{dt}$ when $V = 8$

$V = 8 \Rightarrow \frac{1}{3}\pi r^2 h = 8$. Since $h = 3r$, we have:

$$\frac{1}{3}\pi r^2(3r) = 8 \Rightarrow \pi r^3 = 8 \Rightarrow r^3 = \frac{8}{\pi} \Rightarrow r = \frac{2}{\sqrt[3]{\pi}}$$

$$\left(\& \ h = \frac{6}{\sqrt[3]{\pi}}\right)$$

Since $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(3r) = \pi r^3$

$\Rightarrow \frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt} = 3\pi r^2 \cdot \frac{1}{3}\frac{dh}{dt} \rightarrow \frac{dV}{dt} = \pi r^2 \frac{dh}{dt} \xrightarrow{\text{(at snapshot)}}$

$-2 = \pi\left(\frac{2}{\sqrt[3]{\pi}}\right)^2 \frac{dh}{dt}$

$\Rightarrow \boxed{\frac{dh}{dt} = -\frac{1}{2\sqrt[3]{\pi}} \text{ ft/s}}$

⑭ a) $f(x) = \begin{cases} (x-1)^2 & \text{for } x<1 \\ ax+b & \text{for } 1 \leq x \leq 4 \\ \sqrt{2x+1} & \text{for } 4<x \end{cases}$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} (x-1)^2 = 0 \Rightarrow \text{Need } f(1)=0 \Rightarrow \text{Need } a(1)+b=0$$

$$\Rightarrow a+b=0$$

Also, $\lim_{x\to 4^+} f(x) = \lim_{x\to 4} \sqrt{2x+1} = \sqrt{9} = 3 \Rightarrow$ Need $f(4)=3$

$$\Rightarrow \text{Need } a(4)+b=3$$

$$\Rightarrow 4a+b=3$$

So $\left.\begin{array}{l} a+b=0 \\ \& \ 4a+b=3 \end{array}\right\} \Rightarrow -3a=-3 \Rightarrow a=1 \Rightarrow b=-1$

and if $a=1$ & $b=-1$ $f(x)$ will be continuous at $x=1$ and $x=4$. Hence, it will be continuous everywhere, as it was already continuous away from these points.

b) $\lim_{x\to 0} x\csc(2x) = \lim_{x\to 0} \frac{x}{\sin(2x)} = \lim_{x\to 0} \frac{1}{2} \cdot \frac{2x}{\sin(2x)} = \lim_{x\to 0} \frac{1}{2} \cdot \lim_{x\to 0} \frac{2x}{\sin(2x)}$

$$= \frac{1}{2} \cdot 1 = \boxed{\frac{1}{2}}$$

⑮ a) $\int \frac{1000}{(10+t)^2}\,dt$ let $u=10+t$

$du = dt$

$$\int \frac{1000}{u^2} \cdot du = 1000\int u^{-2}\,du = 1000 \cdot \frac{u^{-1}}{-1} + C = \frac{-1000}{u} + C$$

$$\Rightarrow h(t) = \frac{-1000}{10+t} + C$$

(Balloon starts at ground) $\Rightarrow$ Since $h(0)=0$, we have $-\frac{1000}{10+0} + C = 0 \Rightarrow -100+C=0$

$$\Rightarrow C=100$$

$$\Rightarrow h(t) = \frac{-1000}{10+t} + 100$$

b) As $t \to \infty$, $\frac{-1000}{10+t} \to 0$, so $h(t) \to 100$. Alternatively, $h(t) = \frac{-1000+100(10+t)}{10+t}$

$\Rightarrow h(t) = \frac{100t}{10+t} \Rightarrow \lim_{t\to\infty} h(t) = \lim_{t\to\infty} \frac{10}{\frac{10}{t}+1} = \frac{10}{0+1} = \boxed{10}$