MATH 201 FALL 2009 FINAL SOLUTIONS

PART 1

① a) $y = \underbrace{4x^3}_{f}\underbrace{(2x+1)^2}_{g}$, $f' = 12x^2$, $g' = 2(2x+1)(2) = 8x+4$

$\Rightarrow y' = fg' + gf' = (4x^3)(8x+4) + (2x+1)^2(12x^2)$

$= 4x^2(2x+1)(4x + 3(2x+1))$

$= 4x^2(2x+1)(10x+3)$

b) $y = \dfrac{2x}{\sqrt{x+8}}$ $\begin{matrix}\leftarrow f\\ \leftarrow g\end{matrix}$, $f' = 2$, $g' = \frac{1}{2}(x+8)^{-1/2}(1) = \dfrac{1}{2\sqrt{x+8}}$

$\Rightarrow y' = \dfrac{gf' - fg'}{g^2} = \dfrac{\sqrt{x+8}\,(2) - 2x\left(\frac{1}{2\sqrt{x+8}}\right)}{\left(\sqrt{x+8}\right)^2}$

$= \dfrac{\dfrac{2(x+8)}{\sqrt{x+8}} - \dfrac{x}{\sqrt{x+8}}}{x+8} = \dfrac{2x+16-x}{(x+8)\sqrt{x+8}} = \boxed{\dfrac{x+16}{(x+8)^{3/2}}}$

c) $\frac{d}{dx}\left[y\sin x + x^2y^2 = 2x\right]$

$\left(y\cdot\cos x + \sin x\cdot\frac{dy}{dx}\right) + \left(x^2\cdot 2y\frac{dy}{dx} + y^2\cdot 2x\right) = 2$

$= \frac{dy}{dx}(\sin x + 2x^2y) + y\cos x + 2xy^2 = 2$

$\Rightarrow \boxed{\dfrac{dy}{dx} = \dfrac{2 - y\cos x - 2xy^2}{\sin x + 2x^2y}}$

d) $y = \tan^2(x^3+2) = \left(\tan(x^3+2)\right)^2$

$\Rightarrow y' = 2(\tan(x^3+2))^1(\sec^2(x^3+2))\cdot 3x^2$

$= 6x^2\tan(x^3+2)\cdot\sec^2(x^3+2)$

② a) $\int \frac{4x^2-4x+1}{x^{3/2}}\,dx = \int (4x^{1/2} - 4x^{-1/2} + x^{-3/2})\,dx$

$= \frac{4x^{3/2}}{3/2} - \frac{4x^{1/2}}{1/2} + \frac{x^{-1/2}}{-1/2} + C =$

$= \boxed{\frac{8}{3}x^{3/2} - 8x^{1/2} - 2x^{-1/2} + C}$

b) $\int \sin x \cos x\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\sin x)^2}{2} + C$

$u = \sin x$

$du = \cos x\,dx$

$= \boxed{\frac{\sin^2 x}{2} + C}$

c) $\int \frac{3x^3}{\sqrt{x^4+1}}\,dx = 3\int \frac{x^3\,dx}{\sqrt{x^4+1}}$

Let $u = x^4+1$

$\Rightarrow du = 4x^3\,dx \Rightarrow \frac{du}{4} = x^3\,dx$

$\to 3\int \frac{1}{\sqrt{u}} \cdot \frac{du}{4} = \frac{3}{4} \cdot \frac{u^{1/2}}{1/2} = \frac{3}{2}u^{1/2} + C$

$= \boxed{\frac{3}{2}\sqrt{x^4+1} + C}$

d) $\int_0^{\sqrt{\pi/4}} 4x\sec^2(x^2)\,dx$ ~~Let~~ $u = x^2$

$du = 2x\,dx \Rightarrow \frac{du}{2} = x\,dx$

$\to 4\int_0^{\pi/4} \sec^2(u) \cdot \frac{du}{2} = 2\tan u\Big]_0^{\pi/4} = 2(1-0) = \boxed{2}$

③ a) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

b) $f(x) = x^2+2x \Rightarrow f(x+h) = (x+h)^2 + 2(x+h) =$

$= x^2+2xh+h^2+2x+2h$

$\Rightarrow \frac{f(x+h)-f(x)}{h} = \frac{(x^2+2xh+h^2+2x+2h)-(x^2+2x)}{h} =$

$$= \frac{2xh+h^2+2h}{h} = \frac{h(2x+h+2)}{h} = 2x+h+2, \ (h \neq 0)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} (2x+h+2) = 2x+0+2 = \boxed{2x+2}$$

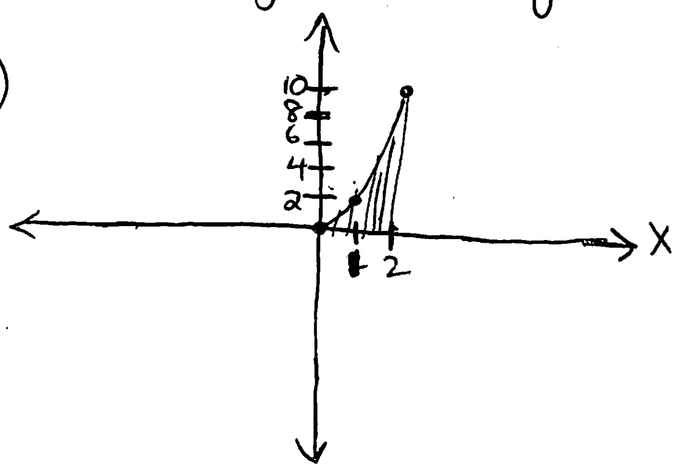
c) $y = f(x) = x^2+2x$ at $(1,3)$

<u>Point</u>: $(1,3)$

<u>m_{tan}</u>: $f'(1) = 2(1)+2 = 2+2 = 4$

$\rightarrow$ Tangent Line: $y-3 = 4(x-1) \Rightarrow \boxed{y = 4x-1}$

④



<u>Note</u>: $x^3+x \geq 0$ for $0 \leq x \leq 2$

$\Rightarrow$ Area is given by

$$\int_0^2 (x^3+x)\,dx = \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^2$$

$$= (4+2) - 0 = \boxed{6}$$

⑤ $f(x) = \dfrac{x^2-9}{(x-1)^2}$, $f'(x) = \dfrac{2(9-x)}{(x-1)^3}$, $f''(x) = \dfrac{4(x-13)}{(x-1)^4}$

A) $\boxed{\text{Intercepts}}$

<u>Y-intercept</u> $(x=0)$

$f(0) = \frac{-9}{1} = -9 \Rightarrow (0,-9)$

<u>X-intercept(s)</u> $(y=0)$

$f(x) = 0 \Rightarrow x^2-9 = 0 \Rightarrow x = 3$ or $x = -3 \Rightarrow (3,0) \, \& \, (-3,0)$

B) $\boxed{\text{Asymptotes}}$

<u>Vertical</u>

$\lim_{x \to 1} f(x) = -\infty$

$\Rightarrow x = 1$ is a V.A.

<u>Horizontal</u>

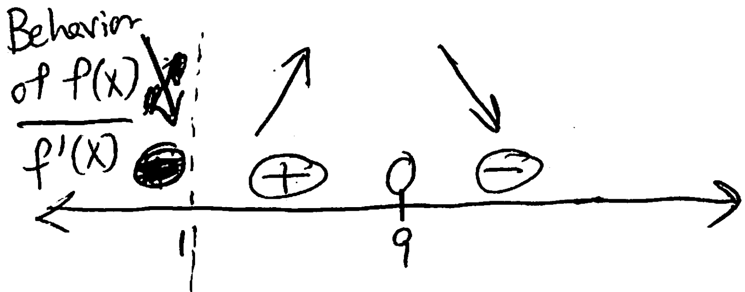
$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2-9}{x^2-2x+1} = \lim_{x \to \infty} \frac{1-\frac{9}{x^2}}{1-\frac{2x}{x^2}+\frac{1}{x^2}} = 1$$

& $\lim_{x\to-\infty} f(x) = 1$ similarly

$\Rightarrow y=1$ is an H.A.

c) $\boxed{\text{Local Maxima/Minima}}$

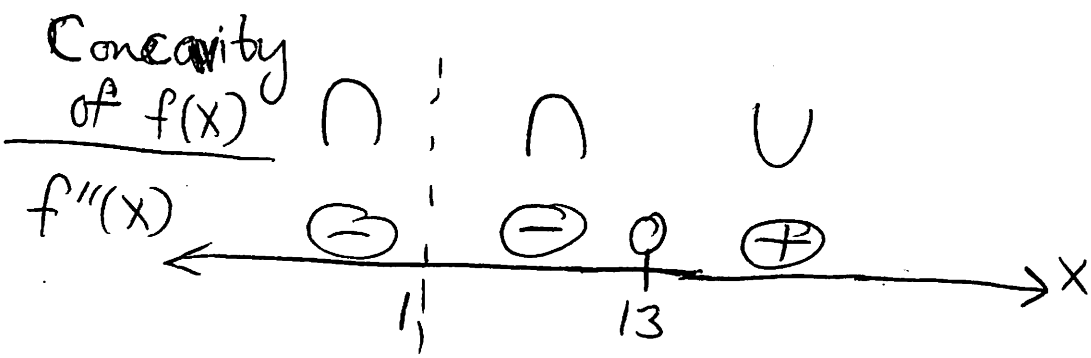
$$f'(x)=0 \Rightarrow 2(9-x)=0 \Rightarrow x=9$$



Since $f(9) = \frac{72}{64} = \frac{9}{8} = 1.125 \Rightarrow (9, 1.125)$ is a local max

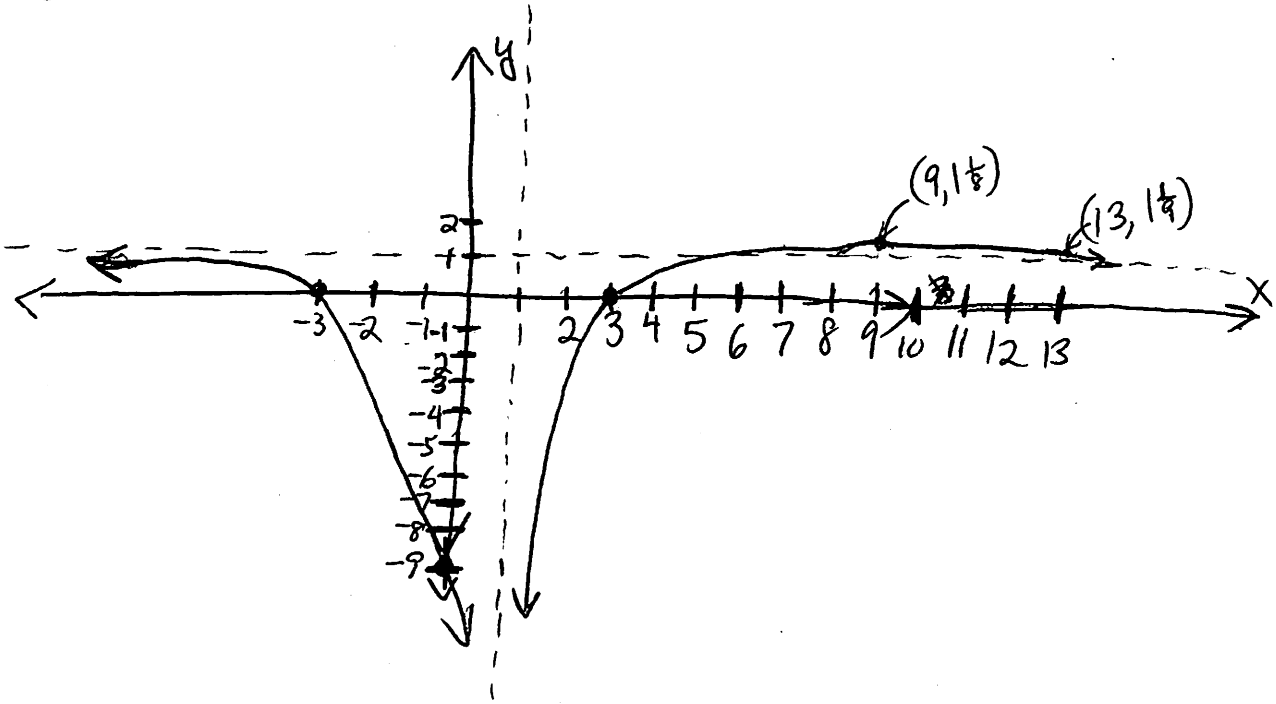
D) $\boxed{\text{Intervals of Concavity \& Inflection Point(s)}}$

$$f''(x)=0 \Rightarrow 4(x-13)=0 \Rightarrow x=13$$



Since $f(13) = \frac{160}{144} = \frac{80}{72} = \frac{10}{9} = 1\frac{1}{9}$ and there is a change in concavity around $x=13$, $(13, 1\frac{1}{9})$ is an inflection point of $f(x)$.

E) $\boxed{\text{SKETCH}}$



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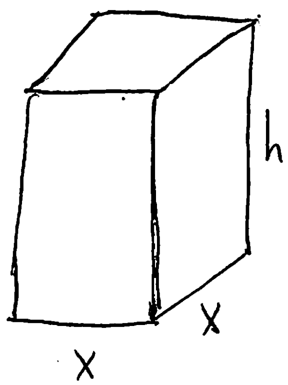


$SA = x^2 + 4xh$ (no top)

Given: $SA = 300$

Maximize: $V = x^2 h$

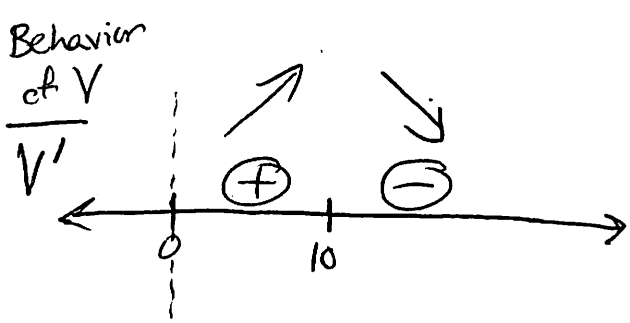
$$x^2 + 4xh = 300 \Rightarrow h = \frac{300 - x^2}{4x}$$

$$\Rightarrow V = x^2\left(\frac{300 - x^2}{4x}\right) = \frac{300x - x^3}{4} = \frac{1}{4}(300x - x^3)$$

$$V' = \frac{1}{4}(300 - 3x^2) \Rightarrow V' = 0 \Rightarrow 300 - 3x^2 = 0$$

$$\Rightarrow x = 10 \text{ or } \cancel{x = -10}$$

reject



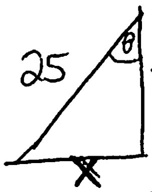
So $x = 10$ puts V at a maximum

So $\underline{x = 10}$ and $\underline{h} = \frac{300 - (10)^2}{4(10)} = \underline{5}$ maximize V.

$\Rightarrow 2(7)(2)+2(24)\frac{dy}{dt}=2(25)(0)$

$\Rightarrow 28+56\frac{dy}{dt}=0 \Rightarrow \frac{dy}{dt}=-\frac{1}{2}$ ft/s

b)



Know: $\sin\theta = \frac{x}{25} = \frac{1}{25}x$

Want: $\frac{d\theta}{dt}$ when $x=7$

$\frac{d}{dt}\left[\sin\theta = \frac{1}{25}x\right]$

$\cos\theta \frac{d\theta}{dt} = \frac{1}{25}\cdot\frac{dx}{dt}$

$\cos\theta$ at snapshot is $\frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{24}{25}$

So $\frac{24}{25}\frac{d\theta}{dt} = \frac{1}{25}\cdot 2 \Rightarrow \frac{d\theta}{dt} = \frac{2}{25}\cdot\frac{25}{24} = \boxed{\frac{1}{12}\text{ radians/second}}$

9. $f(x)=2x^2-\sqrt{x}$ on $[0,1]$

$f'(x)=4x-\frac{1}{2}x^{-1/2} = 4x-\frac{1}{2\sqrt{x}} = \frac{8x\sqrt{x}-1}{2\sqrt{x}}$

$\Rightarrow f'(x)=0$ if $8x\sqrt{x}-1=0 \Rightarrow x^{3/2}=\frac{1}{8} \Rightarrow x=\left(\frac{1}{8}\right)^{2/3}=\frac{1}{4}$

$\Rightarrow$

x	$f(x)$
0	0
1	1 ← absolute maximum on $[0,1]$
1/4	$2(\frac{1}{4})^2-\sqrt{\frac{1}{4}} = \frac{1}{8}-\frac{1}{2} = -\frac{3}{8}$ ← absolute minimum on $[0,1]$

10. a) i) $\lim_{x\to 0}\frac{\tan(2x)}{\tan(3x)} = \lim_{x\to 0}\frac{\sin(2x)}{\cos(2x)}\cdot\frac{\cos(3x)}{\sin(3x)} = \lim_{x\to 0}\frac{2}{3}\frac{\cos(3x)}{\cos(2x)}\cdot\frac{\sin(2x)}{2x}\cdot\frac{3x}{\sin(3x)}$

$=\frac{2}{3}\cdot 1\cdot 1\cdot 1 = \boxed{\frac{2}{3}}$

ii) $\lim_{x\to 1}\frac{x^2-1}{x^2+2x-3} = \lim_{x\to 1}\frac{(x+1)\cancel{(x-1)}}{(x+3)\cancel{(x-1)}} = \frac{1+1}{1+3} = \frac{2}{4} = \boxed{\frac{1}{2}}$

Part 2

(7) a) Let $\cancel{f(x)=\sqrt[3]{x}}$ $f(x)=\sqrt[3]{x} \Rightarrow f'(x)=\dfrac{1}{3(\sqrt[3]{x})^2}$

Then $f(27)=\sqrt[3]{27}=3$

Recall: $L_a(x)=f(a)+f'(a)(x-a)$

Let $a=27$. Then $f(a)=3$ and $f'(a)=\dfrac{1}{27}$

$$\Rightarrow L_{27}(x)=3+\frac{1}{27}(x-27) \Rightarrow L_{27}(26)=3+\frac{1}{27}(26-27)$$

$$=3-\frac{1}{27}=\boxed{2\frac{26}{27}}$$

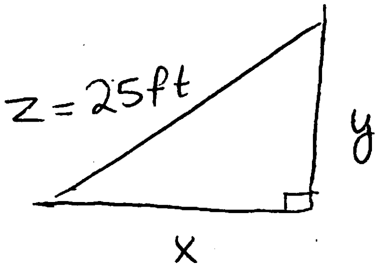
b) L_3 for $f(x)=\dfrac{1}{x}$ on $[1,4]$

$$\Delta x=\frac{b-a}{n}=\frac{4-1}{3}=\frac{3}{3}=1$$

Endpts. to use: $1, 2, 3$

$$\Rightarrow L_3=\underset{\Delta x}{1}\underbrace{\left(\frac{1}{1}\right)}_{f(1)}+\underset{\Delta x}{1}\underbrace{\left(\frac{1}{2}\right)}_{f(2)}+\underset{\Delta x}{1}\underbrace{\left(\frac{1}{3}\right)}_{f(3)}=1+\frac{1}{2}+\frac{1}{3}=\boxed{1\frac{5}{6}}$$

(8)



Know: $x^2+y^2=z^2$

$\dfrac{dx}{dt}=2,\ z=25 \Rightarrow \dfrac{dz}{dt}=0$

a) Snapshot: $x=7 \Rightarrow (7)^2+y^2=(25)^2$

$\Rightarrow y=24$

$$x^2+y^2=z^2 \Rightarrow 2x\frac{dx}{dt}+2y\frac{dy}{dt}=2z\frac{dz}{dt}$$

b) $F(x)=\int_0^{3x}\sqrt{t^2+2t}\,dt \Rightarrow$

$$F'(x)=\sqrt{(3x)^2+2(3x)}\cdot 3=\sqrt{9x^2+6x}\cdot 3$$
$$=3\sqrt{9x^2+6x}$$

$$\Rightarrow F'(1)=3\sqrt{9(1)^2+6(1)}=\boxed{3\sqrt{15}}$$

11) a) $a(t)=\sin(t)+\frac{1}{2}$, $v(0)=0$, $s(0)=0$

$$v(t)=\int\left(\sin(t)+\tfrac{1}{2}\right)dt=-\cos t+\tfrac{1}{2}t+C_1$$

$$v(0)=0 \Rightarrow -\underbrace{\cos(0)}_{1}+\tfrac{1}{2}(0)+C_1=0 \Rightarrow C_1=1$$

$$\Rightarrow \boxed{v(t)=-\cos t+\tfrac{1}{2}t+1}$$

$$\cancel{\Rightarrow s(t)=\int\left(-\cos t+\tfrac{1}{2}t\right)dt=-\sin t+\tfrac{1}{2}\cdot\tfrac{t^2}{2}+C_2}$$

$$\cancel{\Rightarrow s(0)=-\sin(0)}$$

$$\Rightarrow s(t)=\int\left(-\cos t+\tfrac{1}{2}t+1\right)dt=-\sin t+\tfrac{1}{2}\cdot\tfrac{t^2}{2}+t+C_2$$

$$\Rightarrow s(0)=-\sin(0)+\tfrac{1}{4}(0)^2+0+C_2=0 \Rightarrow C_2=0$$

$$\Rightarrow s(t)=\boxed{-\sin t+\tfrac{1}{4}t^2+t}$$

b) $f(x)=x^{-2} \Rightarrow \underset{(\text{on }[1,3])}{f_{ave}}=\frac{1}{3-1}\int_1^3 x^{-2}\,dx=\frac{1}{2}\cdot\frac{x^{-1}}{-1}\Big]_1^3=-\frac{1}{2}\cdot\frac{1}{x}\Big]_1^3$

$$=-\frac{1}{2}\left(\frac{1}{3}-1\right)=-\frac{1}{2}\cdot -\frac{2}{3}$$
$$=\frac{1}{3}$$

$f(x)=x^{-2}=\frac{1}{x^2}$. Where does $\frac{1}{x^2}=\frac{1}{3}$ on $[1,3]$?

$x^2=3 \Rightarrow x=\pm\sqrt{3} \Rightarrow \boxed{x=\sqrt{3}}$ (since on $[1,3]$)